Numerical Investigation of Passively Advection Vector Field in Anisotropic Turbulent Environment

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Introduction-Motivation

• Main problem of fully developed turbulence:
  ★ verification of the basic principles of Kolmogorov-Obukhov theory (KO) within microskopic model
  ★ To investigate possible deviations from KO theory

• Both theoretical and experimental results show deviations from KO theory (see e.g., A.S. Monin and A.M. Yaglom, *Statistical Fluid Mechanics* Vol.2)

• According KO theory (for example) the single-time structure functions

\[
S_N(r) \equiv \langle [v_r(t, \mathbf{x}') - v_r(t, \mathbf{x})]^N \rangle, \quad r = |\mathbf{x}' - \mathbf{x}|
\]  

in the inertial range \((L \ll r \ll l)\) are independent of both: 
\(L\) - external (integral) scale (the first Kolmogorov hypothesis), and 
\(l\) - internal (viscous) scale (the second Kolmogorov hypothesis (KH)).
By dimensional arguments - scale invariant form of SFs:

\[ S_N(r) = \text{const} \times (\bar{\varepsilon}r)^{N/3}, \]  

(2)

\( \bar{\varepsilon} \) - mean dissipation rate.

• But, deviations from the first KH:

\[ S_N(r) = (\bar{\varepsilon}r)^{N/3} R_N(r/L), \]  

(3)

\( R_N(r/L) \) - scaling functions with powerlike behavior in \( r/r_l \ll 1 \):

\[ R_N(r/L) \simeq \text{const} \times (r/L)^{q_N}. \]  

(4)

"anomalous scaling": - singular dependence of SFs on \( L \) \( (L \to \infty) \) + nonlinearity of the exponent \( q_N \).

• Theoretical explanation of AS: strongly developed fluctuations of the dissipation rate - intermittency.
• A suitable method of studying self-similar behavior - renormalization group (RG):
  + Within the RG theory the second KH was proved for structure functions - the existence of IR scaling \( r \gg l \) with "critical dimensions" \( \Delta [S_N] = -N/3 \).
  + But it is not possible to find the scaling function \( R_N(r/L) \) from RG equations.
  + The investigation of \( R_N(r/L) \) in the limit \( r/L \to 0 \) - the operator product expansion (OPE). It leads:
    \[
    R_N(r/r_l) = \sum_F C_F(r/L)^{\Delta F}, \quad r/L \to 0, \quad (5)
    \]
  + critical phenomena - all relevant composite operators - \( \Delta_F > 0 \)
fully developed turbulence based on stochastic NS equation - existence of composite operators with $\Delta_F < 0$ - "dangerous operators": they lead to singular behavior of structure functions in the limit $r/L \to 0$.
- families of dangerous CO with spectrum of critical dimensions unbounded from below exist. An open problem.

- Great progress was achieved within the models of passively advected scalar or vector fields by given Gaussian or non-Gaussian statistics of velocity field: it was shown that the anomalous scaling is related to the existence of dangerous operators.

- It was shown: the behavior of anomalous dimensions is the same for scalar and vector quantity advected by the gaussian statistics of velocity field even in anisotropic situation

- Main reason for present paper: Is it also true for model with non-Gaussian statistics of velocity field?
The Model

• Advection of a passive vector (magnetic) field within kinematic MHD:

\[
\partial_t v_i = \nu_0 \Delta v_i - v_j \partial_j v_i - \partial_i p + f_i, \quad (6)
\]

\[
\partial_t b_i = \nu_0 u_0 \Delta b_i - v_j \partial_j b_i + b_j \partial_j v_i + f^b_i, \quad (7)
\]

- \( v_i = v_i(t, x) \) - \( i \)-th component of the transverse (due to incompressibility) velocity field,
- \( b_i = b_i(t, x) \) - \( i \)-th component of the advected field,
- \( \nu_0 \) - kinematic viscosity coefficient,
- \( \nu_0 u_0 \) - magnetic diffusivity, \( u_0 \) - inverse magnetic Prandtl number,
- \( p \) - pressure,
- \( f_i \) and \( f^b_i \) - random forces,
- statistics of \( f^b_i \) is not important
- $f_i \equiv f_i(x)$ - Gaussian random noise:

$$D_{ij}^f \equiv \langle f_i(x)f_j(x') \rangle = \delta(t - t') \int \frac{d^d k}{(2\pi)^d} R_{ij}(k) d_f(k) e^{i k (x - x')}, \quad (8)$$

where

$$R_{ij}(k) = \left(1 + \alpha_1 \frac{(n \cdot k)^2}{k^2}\right) P_{ij} + \alpha_2 P_{is} n_s n_t P_{tj} \quad (9)$$

with
- $P_{ij} = \delta_{ij} - k_i k_j/k^2$ - common transverse projector,
- $n$ - unit vector of uniaxial anisotropy,
- $d_f(k) = D_0 k^{4-d-2\varepsilon}$,
- $\varepsilon = 0$ - logarithmic theory,
- $\varepsilon = 2$ - real value (gives Kolmogorov dimensions),
- $D_0 = g_0 \nu_0^3$ - amplitude factor
- $d > 2$ - space dimension
- $\alpha_1$ and $\alpha_2$ - anisotropy parameters ($\alpha_{1,2} > -1$)
Field Theoretic Formulation of The Model

- reformulation of the stochastic model into the equivalent field theoretic model with doubled set of fields $\Phi = \{v, b, v', b'\}$ yields

$$S(\Phi) = \frac{1}{2} \int dt_1 d^d x_1 dt_2 d^d x_2 v'_i(t_1, x_1) D^f_{ij}(t_1, x_1; t_2, x_2) v'_j(t_2, x_2)$$

(10)

$$+ \int dt d^d x \ v'_i [-\partial_i v_i - v_j \partial_j v_i + \nu_0 \triangle v_i]$$

$$+ \int dt d^d x \ b'_i [-\partial_i b_i - v_j \partial_j b_i + b_j \partial_j v_i + \nu_0 u_0 \triangle b_i]$$

$$+ \int dt d^d x \ v'_i \nu_0 \left[ \chi_{10} (n \cdot \partial)^2 v_i + \chi_{20} n_i \triangle (n \cdot v) + \chi_{30} n_i (n \cdot \partial)^2(n \cdot v) \right]$$

$$+ \int dt d^d x \ b'_i \nu_0 u_0 \left[ \tau_{10} (n \cdot \partial)^2 b_i + \tau_{20} n_i \triangle (n \cdot b) \right],$$

where $D^f$ is a random force correlator (8).

- the stochastic averaging of random quantities is replaced with functional averages with weight $\exp(S(\Phi))$
• standard field-theoretic analysis:

  ★ dimensional analysis → UV divergent Green functions: \( \langle v'_i v'_j \rangle_{1-ir} \), 
  \( \langle b'_i b'_j \rangle_{1-ir} \)

  ★ renormalization procedure → renormalization group functions and equations for asymptotic (infrared) behavior of correlation functions
Numerical Analysis of the Flow Equations

- Possible scaling regimes - the IR stable fixed points of RG-equations (defined by vanishing of $\beta-$functions)

- The asymptotic behavior of correlation functions is driven by the IR fixed point of RG:

$$\beta_{C_i}(C_j^*) = 0, \quad \Omega_{ij} = \partial_{C_i} \beta_{C_j} |_{C_i = C_j^*},$$

where $C = \{g, u, \chi_1, \chi_2, \chi_3, \tau_1, \tau_2\}$

- But

$$\beta_{C_i} \sim \int_{-1}^{1} \frac{P_i(x^2)}{Q_i(x^2)} (1 - x^2)^{d-3/2} dx,$$

where $P_i(x^2)$ and $Q_i(x^2)$ are polynomials of the parameters
• Another possibility for finding IR fixed points → solving of the Gell-Mann-Low (flow) RG equations (in our case seven equations)

\[ t \frac{d\bar{C}_i}{dt} = \beta_{C_i}(\bar{C}_i; \alpha_{1,2}, d), \quad t \in [0, 1] \quad (12) \]

- initial conditions: \( t = 1 \)
- IR fixed point \( C^*: t = 0 \)

• it is convenient to work with autonomic system: \( t = e^{-s} \)

\[ \frac{d\bar{C}_i}{ds} = -\beta_{C_i}(\bar{C}_i; \alpha_{1,2}, d), \quad s \in [0, \infty] \quad (13) \]

- initial conditions: \( s = 0 \)
- IR fixed point \( C^*: s = \infty \)

• the problem was solved by Runge-Kutta method of four order with adaptive choice of the step
• **Theorem:** Let $\alpha$ be a real number and let $P_0(x)$ and $Q(x)$ be polynomials of real variable $x$ such that $d(P_0(x)) \leq d(Q(x))$, where $d(R(x))$ denotes the degree of a polynomial $R(x)$, and $Q(x)$ is nonzero for $x \in [0, 1]$. Then for arbitrary $m \in \mathbb{Z}_0^+$ the following formula holds:

\[
I = \int_0^1 \frac{P_0(x)(1 - x^2)^\alpha}{Q(x)} \, dx = \sum_{i=1}^m \left[ \frac{1}{4(\alpha + i)} \left( \frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)} \right) + \frac{\sqrt{\pi}}{4} \frac{\Gamma(\alpha + i)}{\Gamma(\alpha + i + 1/2)} \left( \frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)} \right) \right] + \int_0^1 \frac{P_m(x)}{Q(x)} (1 - x^2)^{\alpha + n} \, dx,
\]

where

\[
P_i(x) = \frac{P_{i-1}(x) - (A_ix + B_i)Q(x)}{1 - x^2},
\]

\[
A_i = \frac{1}{2} \left( \frac{P_{i-1}(1)}{Q(1)} - \frac{P_{i-1}(-1)}{Q(-1)} \right), \quad B_i = \frac{1}{2} \left( \frac{P_{i-1}(1)}{Q(1)} + \frac{P_{i-1}(-1)}{Q(-1)} \right)
\]

for $i = 1, 2, \ldots, m$. 
Fixed Point and Stability of Scaling Regime

- it was shown that the system has the same solution as in problem of passive scalar with

\[ \tau_2^* = 0 \]  \hspace{1cm} (14)
Anomalous Scaling of Advected Vector Field

- we are interested in behavior of single-time correlation functions of advected field:

\[ B_{N-m,m}(r) \equiv \langle b_{r}^{N-m}(t, \mathbf{x})b_{r}^{m}(t, \mathbf{x}') \rangle, \quad r = |\mathbf{x} - \mathbf{x}'| \]  \hspace{1cm} (15)

where \( b_{r} \) denotes the component of advected field directed along \( r = \mathbf{x} - \mathbf{x}' \)

- existence of stable IR fixed point leads to representation

\[ B_{N-m,m}(r) \simeq \nu_0^{-N/2}l^N(r/l)^{N(1-\varepsilon/3)-\gamma^*_N-\gamma^*_m} R_{N,m}(r/L), \]  \hspace{1cm} (16)

where \( \gamma^*_N \) and \( \gamma^*_m \) are the fixed point values of the anomalous dimensions of the composite operators \( b_{r}^{N-m} \) and \( b_{r}^{m} \), respectively and the scaling functions \( R_{N,m}(r/L) \) remain unknown within the standard RG analysis.

- Estimation of behavior of \( R_{N,m}(r/L) \) in limit \( r/L \rightarrow 0 \) by Operator Product Expansion (OPE). Possible existence of “dangerous operators” with negative critical dimensions:

\[ R_{N,m}(r/L) = \sum_i C_{F_i}(r/L)(r/L)^{\Delta_{F_i}}, \]  \hspace{1cm} (17)
• the main role is played by operators:

\[
F[N, p] = b_{i_1} \cdots b_{i_p} (b \cdot b)^n, \quad N = 2n + p, \tag{18}
\]

• finally

\[
B_{N-m,m}(r) \approx \nu_0^{-N/2} L^N \left( \frac{l}{L} \right)^{N \epsilon/3} \left( \frac{r}{l} \right)^{-\gamma^*_N - \gamma^*_m} \left( \frac{r}{L} \right)^{\gamma^*_N} \\
\approx r^{-\gamma^*_N - \gamma^*_m + \gamma^*_N}, \tag{19}
\]

• the hierarchy properties of \( \gamma \) functions leads to:

\[
B_{N-m,m}(r) \sim r^{\gamma^*[N,0] - \gamma^*[N-m,0] - \gamma^*[m,0]}, \tag{20}
\]

which holds for even values of \( N \) and \( m \),

\[
B_{N-m,m}(r) \sim r^{\gamma^*[N,0] - \gamma^*[N-m,1] - \gamma^*[m,1]}, \tag{21}
\]

which is valid for even value of \( N \) and odd value of \( m \), and

\[
B_{N-m,m}(r) \sim r^{\gamma^*[N,1] - \gamma^*[N-m,0] - \gamma^*[m,1]}, \tag{22}
\]

for odd values of \( N \) and \( m \). The fourth possibility, namely, odd \( N \) and even \( m \) is equivalent to the last case.
• result: $\gamma_{N,p}^*$ for our vector model are the same as for the model of passive scalar advection.
\[ \gamma^*[4,p]/\varepsilon, \alpha_2=\alpha_1, d=3, \varepsilon=2 \]
Conclusions

• The model of anisotropic advection of vector field by the N-S velocity field was studied

• The renormalization of the model was done

• The system of nonlinear Gell-Mann-Low equations was solved numerically

• The anomalous scaling of the model was investigated

• It was shown that although the model is more complicated than the corresponding model of passive scalar advection the final inertial range behavior of the correlations functions is the same