APPLICATIONS OF COMPUTATIONAL ALGEBRA TO STUDYING POLYNOMIAL SYSTEMS OF DIFFERENTIAL EQUATIONS

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• An approach to solving systems of polynomials using modular arithmetics

• Applications of the approach and other algorithms of computational algebra to ODEs

Reference:
One of problems frequently arising in studies of various phenomena in physical, technical and other sciences is the problem of solution of system of polynomials

\[ f_1(x_1, \ldots, x_n) = 0, \]
\[ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \]
\[ f_m(x_1, \ldots, x_n) = 0. \]  \hspace{1cm} (1)

In this paper we discuss the difficulty of this problem and describe an approach to solve system (1) using calculations modulo a prime number. The method works especially efficiently in the case when \( f_1, \ldots, f_m \) are homogeneous or quasi-homogeneous polynomials.
Example. Let us find the variety in $\mathbb{C}^3$ of the ideal $I = \langle f_1, f_2, f_3, f_4 \rangle$, where

$$
\begin{align*}
  f_1 &= y^2 + x + z - 1 \\
  f_2 &= x^2 + 2yx + 2zx - 4x - 3y + 2yz - 3z + 3 \\
  f_3 &= z^2 + x + y - 1 \\
  f_4 &= 2x^3 + 6zx^2 - 5x^2 - 4x - 7y + 4yz - 7z + 7,
\end{align*}
$$

that is, the solution set of the system $f_1 = 0, f_2 = 0, f_3 = 0, f_4 = 0$. Under the lexicographic ordering with $x > y > z$ a Gröbner basis for $I$ is $G = \{g_1, g_2, g_3\}$, where $g_1 = x + y + z^2 - 1$, $g_2 = y^2 - y - z^2 + z$, and $g_3 = z^4 - 2z^3 + z^2$. Thus the system is equivalent to the system

$$
\begin{align*}
  x + y + z^2 - 1 &= 0 \\
  y^2 - y - z^2 + z &= 0 \\
  z^4 - 2z^3 + z^2 &= 0.
\end{align*}
$$

System (2) is readily solved.

$$
V = V(f_1, f_2, f_3, f_4) = \{(-1, 1, 1), (0, 0, 1), (0, 1, 0), (1, 0, 0)\} \subset \mathbb{C}^3.
$$

This method ALWAYS works when the set of solution is finite: compute a Gröbner basis with respect to a lexicographic order, the basis MUST be triangular. The obtained system is trivial to solve (maybe just numerically).
Thus, in the case of the finite number of solutions theoretically Gröbner basis computation with respect to the lexicographic order provides complete solution to the problem. However practically very often calculation of Gröbner basis, especially with respect to lexicographic orders, faces tremendous computational difficulties. During the execution of the algorithms the size of the intermediate polynomials grows exponentially.

This notorious computational difficulty of the Gröbner basis calculations over $\mathbb{Q}$ is an essential obstacle for using the Gröbner basis theory for the real world applications. However the calculation can be drastically simplified using modular arithmetic. The idea to use modular arithmetic for studying problems where the coefficient growth is significant goes back to Borosh and Fraenkel (1966). Modular approach to Gröbner basis calculations has been studied by E.Arnold, G.L.Ebert, H.-G.Grabe, F.Pauer, F.Winkler and others. It appears, for the first time the modular approach was successfully applied for computations of normal forms and studying limit cycles by V.Edneral (1997).

To perform modular computations we choose a prime number $p$ and do all calculations modulo $p$, that is, in the finite field of the characteristic $p$ (the field $\mathbb{Z}_p = \mathbb{Z}/p$). At first glance it seems that the obtained result has nothing to do with our original problem. However, the modular calculations still keep essential information on our original system and it is often possible to extract this information from the result of calculations in $\mathbb{Z}_p$ and to obtain the exact solution of system (1) over the field of rational numbers.
We now describe the approach to solve system (1) using the modular arithmetic. We remind that the variety of the ideal \( I = \langle f_1, \ldots, f_m \rangle \subset k[x_1, \ldots, x_n] \) in \( k^n \), denoted \( V(I) \), is the zero set of all polynomials of \( I \),

\[
V(I) = \{ A = (a_1, \ldots, a_n) \in k^n | f(A) = 0 \text{ for all } f \in I \},
\]

where here and below \( k = \mathbb{Q}, \mathbb{R}, \mathbb{C} \) or \( \mathbb{Z}_p \). The situation when the variety of a polynomial ideal consists of finite number of points arises very rarely. In generic case the variety consists of infinitely many points, so, generally speaking "to solve" system (1) means to find a decomposition of the variety of the ideal into irreducible components, that is to represent \( V \) as a union of irreducible components, \( V = V_1 \cup \cdots \cup V_m \), where each \( V_i \) is irreducible. For example, \( J = \langle xy, xz \rangle = \langle x \rangle \cup \langle y, z \rangle \), that is, the variety of \( J \) is the union of the plane \( x = 0 \) and the line \( y = z = 0 \).

The radical of \( I \) denoted by \( \sqrt{I} \) is an intersection of prime ideals, \( \sqrt{I} = \cap_{j=1}^{s} Q_j \), where \( V_i (i = 1, \ldots, s) \) is the variety of \( Q_i \). \( Q_i \) are called the minimal associate primes of \( I \).

There are 3 algorithms for irreducible decompositions (computation of minimal associate primes), all implemented in Singular:
– Characteristic sets method (minAssChar)
To perform the rational reconstruction, that is to reconstruct \( p/s \in \mathbb{Q} \) given its image \( t \in \mathbb{Z}/m \), we use the following algorithm by Wang P S, Guy M J, and Davenport J H (1982)

**Step 1.** \( u = (u_1, u_2, u_3) := (1, 0, m), \ v = (v_1, v_2, v_3) := (1, 0, c) \)

**Step 2.** While \( \sqrt{m/2} \leq v_3 \) do \( \{ q := \lfloor u_3/v_3 \rfloor, r := u - qv, u := v, \ v := r \} \)

**Step 3.** If \( |v_2| \geq \sqrt{m/2} \) then error()

**Step 4.** Return \( v_3, v_2 \)

Given an integer number \( c \) and a natural number \( m \) the algorithm produces integers \( v_3 \) and \( v_2 \) such that \( v_3/v_2 \equiv c \mod m \).

**Algorithm for the intersection of two ideals** \( I = \langle f_1, \ldots, f_u \rangle \) and \( J = \langle g_1, \ldots, g_v \rangle \).

**Step 1.** Compute a Gröbner basis \( G' \) of

\[
\langle tf_1(\bar{w}), \ldots, tf_u(\bar{w}), (1 - t)g_1(\bar{w}), \ldots, (1 - t)g_v(\bar{w}) \rangle
\]

in \( k[t, x_1, \ldots, x_n] \) with respect to lex with \( t > x_1 > \cdots > x_n \).

**Step 2.** \( G = G' \cap k[x_1, \ldots, x_n] \).

By the corollary of Hilbert Nullstellensatz called the radical membership test, for a polynomial \( f \) and an ideal \( I = \langle f_1, \ldots, f_m \rangle \) in \( k[x_1, \ldots, x_n] \) \( f \in \sqrt{I} \) if and only if the reduced Gröbner basis of the ideal \( \langle 1 - w f, f_1, \ldots, f_m \rangle \) (here \( w \) is a new variable) is equal to \( \{1\} \).
Decomposition Algorithm

Step 1. Choose a prime number $p$ and compute the minimal associated primes $\tilde{Q}_1, \ldots, \tilde{Q}_s$ of $I$ in $\mathbb{Z}_p[x_1, \ldots, x_n]$.

Step 2. Using the rational reconstruction algorithm lift the ideals $\tilde{Q}_i$ ($i = 1, \ldots, s$) to the ideals $Q_i$ in $\mathbb{Q}[x_1, \ldots, x_n]$ (that is, replace all coefficients of $Q_i$ by the rational numbers computed with the reconstruction algorithm).

Step 3. For each $i = 1, \ldots, s$ using the radical membership test check whether the polynomials $f_1, \ldots, f_s$ are in the radicals of the ideals $Q_i$, that is, whether the reduced Gröbner basis of the ideal $\langle 1 - wf, Q_i \rangle$ is equal to $\{1\}$. If "yes" then go to the step 4, otherwise take another prime $p$ and go to step 1.

Step 4. Compute $Q = \cap_{i=1}^s Q_i \subset \mathbb{Q}[x_1, \ldots, x_n]$.

Step 5. Check that $\sqrt{Q} = \sqrt{I}$, that is, that for any $g \in Q$ the reduced Gröbner basis of the ideal $\langle 1 - wg, I \rangle$ is equal to $\{1\}$ and for any $f \in I$ the reduced Gröbner basis of the ideal $\langle 1 - wf, Q \rangle$ is equal to $\{1\}$. If it is the case then $V(I) = \cup_{i=1}^s V(Q_i)$. If not, then choose another prime $p$ and go to Step 1.

We note that two first steps of the algorithm are well-known and have been used for solution of polynomial systems in many works. The novelty of our approach is that we propose a procedure (Steps 3–5) to check whether the set of solution is complete, that is, no solution is lost using the modular calculations.
It turns out, a theorem of E. Arnold allows to simplify drastically the calculations on Step 5 of the algorithm. To present Arnold’s result we introduce the following notation. Given an ideal $\langle f_1, \ldots, f_m \rangle \subset \mathbb{Q}[x_1, \ldots, x_n]$ we scale appropriately so that each $f_j$ is in $\mathbb{Z}[x_1, \ldots, x_n]$ and each $f_j$ is primitive. We consider the ideal $I_p = \langle \bar{f}_1, \ldots, \bar{f}_m \rangle$, where $\bar{f}_i \equiv f_i \mod p$ ($i = 1, \ldots, m$). For a fixed term order and a given set $F$ of polynomials of $\mathbb{Q}[x_1, \ldots, x_n]$ we denote by $L_p(F)$ the set of the leading power products of $F$.

Theorem. (E. Arnold) Let $I$ be a homogeneous ideal in $\mathbb{Q}[x_1, \ldots, x_n]$, $G_p$ be a reduced Gröbner basis of the ideal $I_p$ and $G \subset \mathbb{Q}[x_1, \ldots, x_n]$ be a set of polynomials such that $L_p(G) = L_p(G_p)$, $G$ is a Gröbner basis for the ideal that it generates, $\langle G \rangle$, and $I \subset \langle G \rangle$. Then $I = \langle G \rangle$.

As an immediate corollary of the theorem we see that if for some prime number $p$ a Gröbner basis of the ideal $I_p$ is $\{1\}$, then any reduced Gröbner basis of the ideal $I$ is $\{1\}$ as well. Therefore, on Step 5 of the algorithm instead of computing over the field $\mathbb{Q}$, we can perform computations over $\mathbb{Z}_p$. If for some $p$ for any $g \in \mathbb{Q}$ the Gröbner basis of $\langle 1 - wg, I \rangle \subset \mathbb{Z}_p[x_1, \ldots, x_n]$ is $\{1\}$ and for any $f \in I$ the reduced Gröbner basis of the ideal $\langle 1 - wf, Q \rangle \subset \mathbb{Z}_p[x_1, \ldots, x_n]$ is $\{1\}$, then $V(I) = \bigcup_{i=1}^{s} V(Q_i)$. 
APPLICATIONS TO DIFFERENTIAL EQUATIONS

Consider the polynomial system

\[ \dot{u} = -v + \sum_{j+l=2}^{n} \alpha_{jl} u^j v^l, \quad \dot{v} = u + \sum_{j+l=2}^{n} \beta_{jl} u^j v^l \]
Complexification: $x = u + iv$  \hspace{1cm} (\bar{x} = u - iv)

\[
\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} \bar{x}^q)
\]

\[
\dot{\bar{x}} = -i(\bar{x} - \sum_{p+q=1}^{n-1} \bar{a}_{pq} \bar{x}^{p+1} x^q)
\]

If $b_{qp} = \bar{a}_{pq}$, $y = \bar{x}$ then from (5) we obtain the "real" system.

\[
\dot{x} = i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -i(y - \sum_{p+q=1}^{n-1} b_{qp} y^q x^p)
\]
The Poncaré-Lyapunov normal form of (5) is

\[ \dot{x}_1 = ix_1 + \sum_{k=1}^{\infty} X_{2k+1} x_1^{k+1} y_1^k, \quad \dot{y}_1 = -iy_1 + \sum_{k=1}^{\infty} Y_{2k+1} x_1^{k} y_1^{k+1}. \] (6)

\(X_{2k+1}, Y_{2k+1}\) are polynomials in the coefficients of (5). \(g_k = X_{2k+1} - Y_{2k+1}\) is called the \(k\)-th focus quantity of system (5). If \(g_k = 0\) for all \(k = 1, 2, \ldots\) then (5) is integrable and the corresponding real system has a center at the origin.

If \(X_{2k+1} = Y_{2k+1} = 0\) for all \(k = 1, 2, \ldots\) then (5) is linearizable and the corresponding real system has an isochronous center at the origin.

The problem of linearizability for system

\[ \dot{x} = p(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q), \quad \dot{y} = -q(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^p) \] (7)

(7) is the problem to decide if it can be transformed to the linear system

\[ \dot{z}_1 = pz_1, \quad \dot{z}_2 = -qz_2 \] (8)

by means of a change of the phase variables

\[ z_1 = x + \sum_{m+j=2}^{\infty} u_{m-1,j}(a, b)x^m y^j, \quad z_2 = y + \sum_{m+j=2}^{\infty} u_{m,j-1}(a, b)x^m y^j. \] (9)
If such transformation exists we say that the system is linearizable, it also means that the normal form of the system is its linear part, that is all resonant coefficients in its normal form vanish. The problem of linearizability has been intensively investigated in recent years.

We obtain necessary and sufficient conditions for linearizability of the eight-parameter cubic system

\[
\begin{align*}
\dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2, \\
\dot{y} &= -2y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2.
\end{align*}
\]

(10)

The necessary and sufficient conditions for integrability of the system has been obtained by Fronville, Sadovskii and Žoładek.

For system (10) we computed the first five pairs of the resonant coefficients of the normal form (we call them the linearizability quantities) \(X_{2,1}, Y_{2,1}, X_{4,2}, Y_{4,2}, \ldots, X_{10,5}, Y_{10,5}\). The polynomials are very long, so we do not present them here. We guess that the linearizability variety of system (10) is equal to the variety of the ideal \(I_5 = \langle X_{2,1}, Y_{2,1}, X_{4,2}, Y_{4,2}, \ldots, X_{10,5}, Y_{10,5}\rangle\). To prove this we first look for the irreducible decomposition of the \(\mathbb{V}(I_5)\).

Using the routine \textit{minAssGTZ} of SINGULAR, and performing the computations in the field of characteristic 32002 we find that the primes are

\[
\begin{align*}
\tilde{Q}_1 &= \langle a_{10} - 2b_{10}, b_{10}^2 + 16001b_{01}b_{2,-1} - a_{01}b_{2,-1}, a_{01}b_{10} - 16001a_{-12}b_{2,-1}, b_{01}a_{01} + 2a_{01}^2 + b_{10}a_{-12}\rangle; \\
\tilde{Q}_2 &= \langle b_{01} + a_{01}, a_{10} - 7996b_{10}, b_{10}^2 + 2743a_{01}b_{2,-1}, a_{01}b_{10} + 1279a_{-12}b_{2,-1}, a_{01}^2 - 12811b_{10}a_{-12}\rangle; \\
\tilde{Q}_3 &= \langle b_{01} - 3999a_{01}, a_{10} + b_{10}, b_{10}^2 - 5999a_{01}b_{2,-1}, a_{01}b_{10} - 4a_{-12}b_{2,-1}, a_{01}^2 + 15364b_{10}a_{-12}\rangle; \\
\tilde{Q}_4 &= \langle b_{01} + 4a_{01}, a_{10} - 7999b_{10}, b_{10}^2 - 2a_{01}b_{2,-1}, a_{01}b_{10} - 6401a_{-12}b_{2,-1}, a_{01}^2 + 12801b_{10}a_{-12}\rangle;
\end{align*}
\]
\[ \tilde{Q}_5 = \langle a_{-12}, a_{01}, b_{01} \rangle; \]
\[ \tilde{Q}_6 = \langle a_{-12}, b_{10}, b_{01} + 2a_{01} \rangle; \]
\[ \tilde{Q}_7 = \langle a_{-12}, b_{01} - 10667a_{01}, a_{10} + 3b_{10}, b_{10}^2 - 10668a_{01}b_{2,-1} \rangle; \]
\[ \tilde{Q}_8 = \langle a_{-12}, a_{01}, a_{10}b_{10} - b_{10}^2 + 16001b_{01}b_{2,-1} \rangle; \]
\[ \tilde{Q}_9 = \langle b_{2,-1}, a_{01}, b_{01}, a_{10} + 2b_{10} \rangle; \]
\[ \tilde{Q}_{10} = \langle b_{2,-1}, a_{01}, b_{01}, a_{10} - 2b_{10} \rangle; \]
\[ \tilde{Q}_{11} = \langle b_{2,-1}, a_{-12}, b_{01} + a_{01}, a_{10} + b_{10} \rangle; \]
\[ \tilde{Q}_{12} = \langle b_{2,-1}, a_{01}, a_{10} - b_{10} \rangle; \]
\[ \tilde{Q}_{13} = \langle b_{2,-1}, b_{10} \rangle; \]

Then, using the rational reconstruction algorithm we obtain the ideals \( Q_k \) \((k = 1, \ldots, 13)\) given in the statement of Theorem 1. Working now in the field of characteristic 0 we check that \( \langle 1 - wX_{2m,m}, Q_k \rangle = \langle 1 \rangle \) and \( \langle 1 - wY_{2m,m}, Q_k \rangle = \langle 1 \rangle \) for all \( k = 1, \ldots, 13 \) and \( m = 1, \ldots, 5 \). That means \( \mathbb{V}(Q_k) \) \((k = 1, \ldots, 13)\) are subvarieties of \( \mathbb{V}(I_5) \) and we can pass to Step 5 of the algorithm. On this step, again in \( \mathbb{Q}[a, b] \), we compute \( Q = \bigcap_{k=1}^{13} Q_k \).

Now, computing in \( \mathbb{Z}_{32003}[a, b] \) we find that for any \( g \in Q \) the reduced Gröbner basis of the ideal \( \langle 1 - wg, I_5 \rangle \) is equal to \{1\} and for any \( f \in I_5 \) the reduced Gröbner basis of the ideal \( \langle 1 - wf, Q \rangle \) is equal to \{1\}. It follows from E.Arnold’s theorem that the corresponding Gröbner bases are equal to \{1\} in \( \mathbb{Q}[a, b] \) as well. It means that
\[
\mathbb{V}(I_5) = \bigcup_{i=1}^{s} \mathbb{V}(Q_i) = \mathbb{V}(Q).
\]

**Theorem 1.** System (10) is linearizable if and only if its coefficients belongs to the variety of one of the ideals \( Q_1, \ldots, Q_{13} \), where

\[
(1) \quad Q_1 = \langle a_{10} - 2b_{10}, b_{10}^2 - a_{01}b_{2,-1} - \frac{1}{2}b_{01}b_{2,-1}, a_{01}b_{10} + \frac{1}{2}a_{-12}b_{2,-1}, 2a_{01}^2 + a_{01}b_{01} + a_{-12}b_{10} \rangle;
\]
Proof. We have shown that $V(I_5) = \bigcap_{i=1}^{13} V(Q_i)$, that is, the varieties $V(Q_k)$ ($k = 1, \ldots, 13$) define the necessary conditions for linearizability of system (10). To show that they also are the sufficient conditions for linearizability we verify that if the coefficients of the system belong to one of the varieties $V(Q_k)$ ($k = 1, \ldots, 13$) then the corresponding system is linearizable.

We consider in details only Case 12, the other cases are treated similarly. We wish to find a linearization of the second equation of the system (10) of the form $Y = Y(x, y)$, which
satisfies the equation
\[ \frac{\partial Y}{\partial x} P + \frac{\partial Y}{\partial y} Q + 2Y = 0. \] (11)

After the substitution \( x = yz \), (11) becomes
\[ \frac{1}{y} \frac{\partial Y}{\partial z} P + \left( \frac{\partial Y}{\partial y} - \frac{z}{y} \frac{\partial Y}{\partial z} \right) Q + 2Y = 0. \] (12)

Using induction on \( k \), it is not difficult to check that a solution to (12) can be obtained in the form
\[ Y(z, y) = y \left( 1 + \sum_{k=1}^{\infty} g_k(z) y^k \right), \]
where \( g_k(z) \) is a polynomial of degree \( k \). Thus \( Y(x/y, y) \) is a series in \( x \) and \( y \) of the form
\[ Y(x/y, y) = y + \sum_{k+j=1}^{\infty} Y_{k,j} x^k y^j. \]

Then, for \( \psi \) a first integral of (10), the first equation of (10) is linearizable by the substitution \( X = \psi \).

In Case (12), with the conditions \( b_{2,-1} = a_{01} = a_{10} - b_{10} = 0 \) and the substitution \( x = yz \) the system (10) becomes
\[ P(z, y) = yz - b_{10} y^2 z^2 - a_{-12} y^2, \]
\[ Q(z, y) = -2y + b_{10} y^2 z + b_{01} y^2. \]
and one can expand (12) to obtain

\[
\begin{aligned}
3zg_1'(z) - 2g_1(z) + b_{10}z + b_{01} &= 0, & k &= 1; \\
3zg_k'(z) - 2kg_k(z) - g_{k-1}'(z)(2b_{10}z^2 + a_{-12} + b_{01}z) &+ kg_{k-1}(z)(b_{10}z + b_{01}) &= 0, & k &\geq 2.
\end{aligned}
\]

It turns out that \( g_1(z) = \frac{1}{2}b_{01} - b_{10}z \) (setting the integration constant to zero) and \( g_k(z) \) is a polynomial of degree at most \( k - 2 \) for \( k \geq 2 \).
THE CYCLICITY OF CENTER OF FOCUS OF POLYNOMIAL SYSTEMS  
(the local 16th Hilbert problem)

Theorem. (Generalized Bautin’s theorem) If the ideal $B$ of all focus quantities of system

$$
\dot{x} = (x - \sum_{p+q=1}^{n-1} a_{pq}x^{p+1}y^q), \quad \dot{y} = -(y - \sum_{p+q=1}^{n-1} b_{qp}x^qy^p)
$$

is generated by the $m$ first focus quantities, $B = \langle g_1, g_2, \ldots, g_m \rangle$, then at most $m$ limit cycles bifurcate from the origin of the corresponding real system

$$
\dot{u} = \lambda u - v + \sum_{j+l=2}^{n} \alpha_{jl}u^jv^l, \quad \dot{v} = u + \lambda v + \sum_{j+l=2}^{n} \beta_{jl}u^jv^l,
$$

that is the cyclicity of the system is less or equal to $m$. 

The problem of cyclicity has been solved for the quadratic system
\[
\dot{u} = -v + a_{20} u^2 + a_{11} uv - a_{02} v^2, \quad \dot{v} = u + b_{20} u^2 + b_{11} uv + b_{02} v^2,
\]
(13)
or in the complex form
\[
\dot{x} = x - a_{10} x^2 - a_{01} xy - a_{-12} y^2, \quad \dot{y} = -(y - b_{10} xy - b_{01} y^2 - b_{2,-1} x^2).
\]
(14)

**Lemma 1.** The variety of the Bautin ideal of system (14) coincides with the variety of the ideal \( \mathcal{B}_3 = \langle g_{11}, g_{22}, g_{33} \rangle \) and consists of four irreducible components:
1) \( \mathcal{V}(J_1) \), where \( J_1 = \langle 2a_{10} - b_{10}, 2b_{01} - a_{01} \rangle \),
2) \( \mathcal{V}(J_2) \), where \( J_2 = \langle a_{01}, b_{10} \rangle \),
3) \( \mathcal{V}(J_3) \), where \( J_3 = \langle 2a_{01} + b_{01}, a_{10} + 2b_{10}, a_{01}b_{10} - a_{-12}b_{2,-1} \rangle \),
4) \( \mathcal{V}(J_4) = \langle f_1, f_2, f_3, f_4, f_5 \rangle \), where
\[
\begin{align*}
f_1 &= a_{01}^3 b_{2,-1} - a_{-12} b_{2,10}^3, \\
f_2 &= a_{10} a_{01} - b_{01} b_{10}, \\
f_3 &= a_{10}^2 a_{-12} - b_{2,-1} b_{01}^3, \\
f_4 &= a_{10} a_{-12} b_{10}^2 - a_{01}^2 b_{2,-1} b_{01}, \\
f_5 &= a_{10} a_{-12} b_{10} - a_{01} b_{-2,-1} b_{01}^2.
\end{align*}
\]

**Theorem 2 (Bautin).** The cyclicity of the origin of system
\[
\dot{u} = \lambda u - v + \alpha_{20} u^2 + \alpha_{11} uv + \alpha_{02} v^2, \quad \dot{v} = u + \lambda v + \beta_{20} u^2 + \beta_{11} uv + \beta_{02} v^2
\]
equals three.

Proofs were given by Bautin (1952), Žoładek (1994), Yakovenko (1995), Françoise and Yomdin (1997), Han, Hong Zang and Tonghua Zhang (2007).
By Bautin’s theorem in order to prove that the cyclicity of the origin is at most three, it is sufficient to show that $\mathcal{B} = \mathcal{B}_3$. By Hilbert Nullstellensatz, the latter equality holds if $\mathcal{B}_3$ is a radical ideal. Indeed, according to Lemma 1 we have for all $k$

$$g_{kk} |_{\mathcal{V}(\mathcal{B}_3)} \equiv 0. \quad (15)$$

Hence, if $\mathcal{B}_3$ is a radical ideal then (15) and Hilbert Nullstellensatz yield that $g_{kk} \in \mathcal{B}_3$. Thus, to prove that an upper bound for the cyclicity is equal to three it is sufficient to show that $\mathcal{B}_3$ is a radical ideal.

With help of Singular we check that

$$\text{std}(\text{radical}(\mathcal{B}_3)) = \text{std}(\mathcal{B}_3), \quad (16)$$

where the routines \textit{radical} and \textit{std} compute a radical and a Groebner basis of a polynomial ideal, respectively. The equality (16) means that $\mathcal{B}_3$ is a radical ideal. This completes the proof.
However, as rule the Bautin ideal is not a radical one. Very recently we have succeeded to investigate cyclicity of one more system:

\[
\dot{x} = i(x - a_{-12}x^2 - a_{20}x^2\bar{x} - a_{02}\bar{x}^3),
\]


Lemma 2. The variety of the Bautin ideal of system

\[
\dot{x} = i(x - a_{-12}y^2 - a_{20}x^2y - a_{02}y^3),
\]

\[
\dot{y} = -i(y - b_{2,-1}x^2y - b_{20}x^2y - b_{02}y^3).
\]

coincides with the variety of the ideal \( B_6 = \langle g_{11}, g_{22}, \ldots, g_{66} \rangle \) and consists of five irreducible components:

1) \( a_{20}a_{02} - b_{20}b_{02} = a_{-1,2}^2b_{20}^3 - b_{2,-1}^2a_{02}^3 = a_{-1,2}a_{20}b_{20}^2 - b_{2,-1}a_{02}b_{02} = a_{-1,2}a_{20}^2b_{20} - b_{2,-1}a_{02}b_{02} = a_{-1,2}b_{20}^2 - b_{2,-1}b_{02}^3 = 0 \)
2) \( b_{20} = 2a_{02} + b_{02} = a_{20} = a_{-1,2} = 0 \)
3) \( b_{02} = a_{02} = a_{20} + 2b_{20} = b_{2,-1} = 0 \)
4) \( a_{02} + b_{02} = a_{20} + b_{20} = 0 \)
5) \( a_{02} - 3b_{02} = 3a_{20} - b_{20} = 0 \)

Theorem 3. The cyclicity of the origin of system

\[
\dot{x} = i(x - a_{-12}y^2 - a_{20}x^2\bar{x} - a_{02}\bar{x}^3).
\]

equals four.
The idea of the proof. Due to the Generalized Bautin Theorem, in order to prove that the
cyclicity of the origin of system (18) is at most four, it is sufficient to show that for \( k > 6 \)
\[
\begin{align*}
g_{kk} &= g_{22}h_2 + g_{44}h_4 + g_{55}h_5 + g_{66}h_6 \\
(19)
\end{align*}
\]
in \( \mathbb{C}[a, b] \). The first six focus quantities, each reduced modulo the ideal generated by the
previous ones are:
\[
\begin{align*}
g_1 &= g_{33} = 0, \\
g_2 &= -i(a_{20}a_{02} - b_{20}b_{02}), \\
g_4 &= a_{-1,2}^2a_{20}^3 + \frac{8}{3}a_{-1,2}a_{20}^2b_{20} + a_{-1,2}a_{20}b_{20}^2 - \frac{2}{3}a_{-1,2}b_{20}^3 + \frac{2}{3}a_{02}b_{2, -1}^2 - a_{02}b_{02}b_{2, -1}^2 - \frac{8}{3}a_{02}b_{02}b_{2, -1}^2 - b_{02}b_{2, -1}^2, \\
g_5 &= -\frac{7}{8}a_{-1,2}^3a_{20}b_{20}b_{2, -1} - \frac{7}{12}a_{-1,2}a_{20}b_{20}^2b_{2, -1} + \frac{7}{24}a_{-1,2}b_{20}^3b_{2, -1} - \frac{7}{24}a_{-1,2}a_{02}b_{2, -1}^3 + \frac{7}{12}a_{-1,2}a_{02}b_{02}b_{2, -1}^3 + \frac{7}{8}a_{-1,2}a_{02}b_{02}^2b_{2, -1}^3, \\
g_6 &= -5a_{-1,2}^2a_{20}b_{20}b_{02} + \frac{5}{3}a_{-1,2}a_{02}b_{20}^4 - \frac{10}{3}a_{-1,2}b_{20}^2b_{02} - \frac{5}{3}a_{02}b_{20}b_{2, -1}^2 + \frac{10}{3}a_{02}b_{20}b_{02}b_{2, -1}^2 + 5a_{02}^2b_{20}b_{02}b_{2, -1}^2.
\end{align*}
\]

If the ideal \( \mathcal{B}_6 \) were a radical ideal then (19) would be an immediate corollary of the fact that
\( \mathcal{V}(\mathcal{B}) = \mathcal{V}(\langle g_2, g_4, g_5, g_6 \rangle) = \mathcal{V}(\mathcal{B}_6) \). However, the calculations (e.g. with Singular) show
that \( \mathcal{B}_6 \) is not a radical ideal. Nevertheless we are able to show that despite of the ideal is
not radical in the ring of usual polynomial, it is still radical in a polynomial subalgebra, that is
in the ring \( \mathbb{C}[f_1, \ldots, f_t] \), where \( f_1, \ldots, f_t \) are some polynomials which are invariants of the
rotation group of the system of ODE’s.
Subalgebra of the invariants of the rotation group

Let $k$ be a field and $G$ be a group of $n \times n$ matrices with elements in $k$. A polynomial $f \in k[x_1, \ldots, x_n]$ is invariant under $G$ if $f(\bar{w}) = f(A\bar{w})$ for every $A \in G$.

Consider general polynomial system

$$
\frac{dx}{dt} = -\sum_{p+q=0}^m a_{pq} x^{p+1} y^q = P(x, y), \quad \frac{dy}{dt} = \sum_{p+q=0}^m b_{qp} x^q y^{p+1} = Q(x, y).
$$

Consider the group of rotations

$$
x' = e^{-i\varphi} x, \quad y' = e^{i\varphi} y
$$

of the phase space $\mathbb{C}^2$ of (20). Viewing the action of an element of the group as a coordinate transformation, in $(x', y')$ coordinates system (7) has the form

$$
\dot{x'} = -\sum_{p+q=0}^m a(\varphi)_{pq} x'^{p+1} y'^q, \quad \dot{y'} = \sum_{p+q=0}^m b(\varphi)_{qp} x'^q y'^{p+1},
$$

where the coefficients of the transformed system are

$$
a(\varphi)_{pj} = a_{pj} e^{i(p-j)\varphi}, \quad b(\varphi)_{qj} = b_{qj} e^{i(q-j)p\varphi},
$$

for $j = 1, \ldots, \ell$. For any fixed angle $\varphi$ the equations in (22) determine an invertible linear mapping $U_\varphi$ of the space $(a, b)$ of parameters of (20) onto itself.
Example. For the family of systems

\[ \dot{x} = -a_{00}x - a_{11}y - a_{20}x^3, \quad \dot{y} = b_{1,-1}x + b_{00}y + b_{02}y^3 \]  

(23)

equation (22) gives the collection of \( 2\ell = 6 \) equations

\[ a(\varphi)_{00} = a_{00}e^{i(0-0)\varphi} \quad a(\varphi)_{-11} = a_{-11}e^{i(-1-1)\varphi} \quad a(\varphi)_{20} = a_{20}e^{i(2-0)\varphi} \]

\[ b(\varphi)_{00} = b_{00}e^{i(0-0)\varphi} \quad b(\varphi)_{1,-1} = b_{1,-1}e^{i(1-(-1))\varphi} \quad b(\varphi)_{02} = b_{02}e^{i(0-2)\varphi} \]

so that

\[
U_\varphi \cdot (a, b) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & e^{-2i\varphi} & 0 & 0 & 0 & 0 \\
0 & 0 & e^{2i\varphi} & 0 & 0 & 0 \\
0 & 0 & 0 & e^{-2i\varphi} & 0 & 0 \\
0 & 0 & 0 & 0 & e^{2i\varphi} & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
a_{00} \\
a_{-11} \\
a_{20} \\
b_{02} \\
b_{1,-1} \\
b_{00}
\end{pmatrix} = \begin{pmatrix}
a_{00} \\
a_{-11}e^{-2i\varphi} \\
a_{20}e^{2i\varphi} \\
b_{02}e^{-2i\varphi} \\
b_{1,-1}e^{2i\varphi} \\
b_{00}
\end{pmatrix}.
\]

The group \( U = \{U_\varphi : \varphi \in \mathbb{R}\} \) is the rotation group of family (7). A polynomial invariant of the group \( U \) is termed an invariant of the rotation group.
We wish to identify all polynomial invariants of this group action for the system (7):

\[
\frac{dx}{dt} = - \sum_{p+q=0} a_{pq} x^{p+1} y^q = P(x, y), \quad \frac{dy}{dt} = \sum_{p+q=0} b_{qp} x^q y^{p+1} = Q(x, y).
\]

An algorithm for computing a generating set for the subalgebra of invariants of the rotation group (or Lie-invariants as termed by Liu Yi Rong and Li Ji Bin) (R. Laubenbahrer, A. Jarrah and V. Romanovski, J. Symbolic Computation (2003); V. Romanovski, Open Systems and Informational Dynamics (2008)).

Let

\[
H = \langle 1 - w \tilde{\gamma}_1 \cdots \tilde{\gamma}_\ell, \ a_{p_kq_k} - t_k, \ \tilde{\gamma}_k b_{q_kp_k} - \tilde{\gamma}_k t_k \mid k = 1, \ldots, \ell \rangle, \quad (24)
\]

where \( \tilde{\gamma}_k = \gamma^{q_k-p_k} \), \( \tilde{\gamma}_k = 1 \) if \( p_k - q_k \leq 0 \), \( \tilde{\gamma}_k = 1 \), \( \tilde{\gamma}_k = \gamma^{p_k-q_k} \) if \( p_k - q_k > 0 \) and \( \langle h_1, \ldots, h_m \rangle \) denotes the ideal generated by polynomials \( h_1, \ldots, h_m \).

- Compute a Groebner basis \( G_H \) for \( H \) defined by (24) with respect to any elimination order with \( \{ w, \gamma, t_k \} \succ \{ a_{p_kq_k}, b_{q_kp_k} \mid k = 1, \ldots, \ell \} \);
- the set \( G_H \cap k[a, b] \) is a set of binomials whose monomials form a generating set for the subalgebra of all invariants, that is, any other invariant is a polynomial of the monomials.
For the system of our interest

\[
\dot{x} = i(x - a_{-12}y^2 - a_{20}x^2y - a_{02}y^3), \\
\dot{y} = -i(y - b_{2,-1}x^2y - b_{20}x^2y - b_{02}y^3)
\]

we compute a Groebner basis of the ideal

\[\mathcal{J} = \langle 1 - w\gamma^4, a_{-12} - t_1, \gamma^3b_{2,-1} - t_1, a_{20} - t_2, b_{02} - \gamma^2t_2, a_{02} - t_3, \gamma^2b_{20} - t_3 \rangle\]

with respect to the lexicographic order with \( w > \gamma > t_1 > t_2 > t_3 > a_{-12} > a_{20} > a_{02} > b_{20} > b_{02} > b_{2,-1} \) we obtain a list of polynomials and pick up the polynomials that do not depend on \( w, \gamma, t_1, t_2, t_3 \):

\[
\begin{align*}
a_{20}a_{02} - b_{20}b_{02}, & \quad a_{-12}a_{20}b_{20}^2 - a_{02}b_{2,-1}^3b_{02}, \\
a_{-12}a_{20}b_{20} - a_{02}b_{2,-1}^2b_{02}, & \quad a_{20}^2 - a_{02}b_{2,-1} - a_{-12}b_{20}^3, \\
a_{-12}a_{20}^3 - b_{2,-1}^2b_{02}, & \quad a_{20}a_{02} + b_{20}b_{02}, \\
a_{-12}a_{20}b_{20}^2 + a_{02}b_{2,-1}^2b_{02}, & \quad a_{-12}a_{20}b_{20} + a_{02}b_{2,-1}^2b_{02}, \\
a_{02}b_{2,-1}^3b_{02}, & \quad a_{02}b_{2,-1} + a_{-12}b_{20}^3, \\
a_{-12}a_{20}^3 + b_{2,-1}^2b_{02}. & \quad \beta_{-12}b_{20}^3 + b_{2,-1}^2b_{02}.
\end{align*}
\]

The monomials of the binomials form a basis of the invariants:

\[
\begin{align*}
c_1 &= a_{-12}b_{2,-1}, \\
c_2 &= a_{20}b_{02}, \\
c_3 &= a_{02}b_{20}, \\
c_4 &= b_{20}b_{02}, \\
c_5 &= a_{02}^2b_{2,-1}^2, \\
c_6 &= a_{02}b_{2,-1}^2b_{02}, \\
c_7 &= a_{02}b_{2,-1}^2b_{02}, \ldots
\end{align*}
\]

The focus quantities of system (7) belong to the the subalgebra \( \mathbb{C}[c_1, \ldots, s_{15}] \) that is,

\[
g_{kk} = g_{kk}(c_1, \ldots, c_{15}). \tag{25}
\]

Then, we make the substitution \( c_1 = a_{-12}b_{2,-1}, c_2 = a_{20}b_{02}, c_3 = a_{02}b_{20}, c_4 = b_{20}b_{02}, c_5 = a_{02}^2b_{2,-1}^2, c_6 = a_{02}^2b_{2,-1}b_{02}, c_7 = a_{02}b_{2,-1}^2b_{02}, \ldots \) and show that the Bautin ideal is radical ideal in
the new variables. More precisely, consider the ideal
\[ J = \langle c_1 - a_{-12}b_{-1}, c_2 - a_{20}b_{02}, c_3 - a_{02}b_{20}, c_4 - b_{20}b_{02}, c_5 - a_{02}b_{2,-1}, c_6 - a_{02}b_{2,-1}b_{02}, c_7 - a_{02}b_{2,-1}b_{02}^2, c_8 - b_{2,-1}b_{02}^2, c_9 - a_{20}a_{02}, c_{10} - a_{-12}b_{20}, c_{11} - a_{-12}a_{20}b_{20}, c_{-1,2} - a_{-12}a_{20}b_{20}, c_{13} - a_{-12}a_{20}^3 \rangle \] and the corresponding map
\[ F : E(a, b) = A^6 = C^6 \rightarrow A^{13}_C = C^{13}, \]
that is, \( F(a, b) = (a_{-12}b_{-1}, a_{20}b_{02}, a_{02}b_{20}, b_{20}b_{02}, a_{02}b_{2,-1}, \ldots, a_{-12}a_{20}^3) \). Clearly, \( F \) is a morphism of affine algebraic sets. Let \( W \) be the image of \( E(a, b) \) under \( F \) and \( C[c] := C[c_1, \ldots, c_{13}] \). \( F \) induces the \( C \)-algebra homomorphism
\[ F^* : C[c] \rightarrow C[a, b]. \]

Let \( \prec_{(a,b)} \) be an elimination monomial ordering for \( (a, b) \) in the algebra \( C[a, b] \otimes_C C[c] = C[a, b, c] \). Computing the Gröbner basis \( J_G \) of \( J \) with respect to \( \prec_{(a,b)} \), we find that \( J \cap C[c] \) is the ideal \( R \), generated by
\[
\begin{align*}
&c_{11}c_{13} - c_{12}^2, c_{10}c_{13} - c_{11}c_{12}, c_{10}c_{12} - c_{11}^2, c_{6}c_{8} - c_{7}^2, c_{5}c_{8} - c_{6}c_{7}, c_{5}c_{7} - c_{6}^2, c_{4}c_{7}c_{13} - c_{8}c_{9}c_{12}, c_{4}c_{7}c_{12} - c_{8}c_{9}c_{11}, c_{4}c_{7}c_{11} - c_{8}c_{9}c_{10}, c_{4}c_{6}c_{13} - c_{7}c_{9}c_{12}, c_{4}c_{6}c_{12} - c_{7}c_{9}c_{11}, c_{4}c_{6}c_{11} - c_{7}c_{9}c_{10}, c_{4}c_{5}c_{13} - c_{6}c_{9}c_{12}, c_{4}c_{5}c_{12} - c_{6}c_{9}c_{11}, c_{4}c_{5}c_{11} - c_{6}c_{9}c_{10}, c_{3}c_{13} - c_{9}c_{12}, c_{3}c_{12} - c_{9}c_{11}, c_{3}c_{11} - c_{9}c_{10}, c_{3}c_{8} - c_{4}c_{7}, c_{3}c_{7} - c_{4}c_{6}, c_{3}c_{6} - c_{4}c_{5}, c_{2}c_{12} - c_{4}c_{13}, c_{2}c_{11} - c_{4}c_{12}, c_{2}c_{10} - c_{4}c_{11}, c_{2}c_{7} - c_{8}c_{9}, c_{2}c_{6} - c_{7}c_{9}, c_{2}c_{5} - c_{6}c_{9}, c_{2}c_{3} - c_{4}c_{9}, c_{1}c_{3}c_{9} - c_{5}c_{13}, c_{1}c_{4}c_{9} - c_{6}c_{12}, c_{1}c_{4}c_{9} - c_{7}c_{11}, c_{1}c_{4}c_{9} - c_{8}c_{10}, c_{1}c_{3}c_{9} - c_{5}c_{12}, c_{1}c_{3}c_{4} - c_{6}c_{11}, c_{1}c_{3}c_{4} - c_{7}c_{10}, c_{1}c_{3}c_{4} - c_{5}c_{11}, c_{1}c_{3}c_{4} - c_{6}c_{10}, c_{1}c_{3}c_{4} - c_{5}c_{10}, c_{1}c_{2}c_{9} - c_{6}c_{13}, c_{1}c_{2}c_{9} - c_{7}c_{12}, c_{1}c_{2}c_{9} - c_{8}c_{11}, c_{1}c_{2}c_{9} - c_{7}c_{13}, c_{1}c_{2}c_{9} - c_{8}c_{12}, c_{1}c_{2}c_{9} - c_{8}c_{13}.
\end{align*}
\]

\( R \) is the kernel of \( F^* \) (we computed it with the routine preimage of SINGULAR). Let \( C \) be
the subalgebra of $\mathbb{C}[a, b]$, generated by the monomials, corresponding to the components of the map $F$ (that is, by $a_{-12}b_{-1}, a_{20}b_{02}, a_{02}b_{20}$ etc.). Obviously, $C$ is isomorphic to $\mathbb{C}[c]/R$. For a polynomial $f(a, b) \in \mathbb{C}[c_1(a, b), \ldots, c_{13}(a, b)] \subset \mathbb{C}[a, b]$ we denote by $f^F \in \mathbb{C}[c]$ the preimage of $f(a, b)$ under $F^\ast$. Then, $f^F \in \mathbb{C}[c_1(a, b), \ldots, c_{13}(a, b)]$ can be computed via the normal form (or division with remainder), that is $f^F = \text{NF}(f, J_G)$, where $J_G$ is a Gröbner basis of $J$ with respect to an elimination ordering $\prec_{(a, b)}$.

Performing computations, we obtain

$g_{11}^F = 0$, $g_{22}^F = c_9 - c_4$, $g_{33}^F = 0$,
$g_{44}^F = \frac{2}{3} c_5 - c_6 - \frac{8}{3} c_7 - c_8 - \frac{2}{3} c_{10} + c_{11} + \frac{8}{3} c_{12} + c_{13}$,
$g_{55}^F = -\frac{7}{24} c_1 c_5 + \frac{7}{12} c_1 c_6 + \frac{7}{8} c_1 c_7 + \frac{7}{24} c_1 c_{10} - \frac{7}{12} c_1 c_{11} - \frac{7}{8} c_1 c_{12}$,
$g_{66}^F = -\frac{5}{3} c_3 c_5 + \frac{5}{3} c_3 c_{10} + \frac{10}{3} c_4 c_5 + 5 c_4 c_6 - \frac{10}{3} c_4 c_{10} - 5 c_4 c_{11}$.
(The polynomials $g$ are much simple in new coordinates $c_i$; the methods for transformation to such coordinates were developed recently in the theory of invariants of finite groups).

Using some theorems and algorithms from the books:


we show that the ideal $G_6 = \langle g_{11}^F, \ldots, g_{66}^F \rangle$ is a radical ideal in $\mathbb{C}[c]/R$ yielding
for all $k > 6$. It implies that for any $k > 6$ there are polynomials $h_{i,k}$ such that for $c \in \overline{W}$

$$g_{kk}^F = g_{11}^F h_{1,k}^F + g_{22}^F h_{2,k}^F + \cdots + g_{66}^F h_{6,k}^F.$$ 

Going back in the latter equality from the variables $c$ to the variables $(a, b)$ and taking into account that $g_{11} \equiv g_{33} \equiv 0$ we see that

$$g_{kk} = g_{22} h_{2,k} + g_{44} h_{4,k} + g_{55} h_{5,k} + g_{66} h_{6,k}.$$ 

Thus, by Bautin’s Theorem the cyclicity of system (18) is at most four.