An algorithm for symbolic solving systems of partial differential equations

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Laplace-Carson transform (LC)

$$f(X) \mapsto F(P) = P^1 \int_0^\infty e^{-PX} f(X) dX,$$

$X = x_1, \ldots x_n, \quad P = p_1, \ldots p_n, \; P^1 = p_1 \ldots p_n, \quad PX = \langle X, P \rangle,$
$dX = dx_1 \ldots dx_n.$

At that $\eta(X) \mapsto 1.$

PDE system

Consider a system

$$\sum_{k=1}^{K}\sum_{m=0}^{M} a^j_{mk}\frac{\partial^m}{\partial^{m_1}x_1\ldots\partial^{m_n}x_n}u^k(X)=f_j,\quad j=1,\ldots K, \tag{1}$$

where $m_1+\ldots+m_n=m$, $u^k(X)$, $k=1,\ldots,K$, – are unknown functions of $X=x_1,\ldots,x_n$, a^j_{mk} – constants.

Initial conditions

We denote by

$$\Gamma = (\Gamma_1, \ldots, \Gamma_n),\ \beta = (\beta_1, \ldots, \beta_n), \beta_i = 0, \ldots m_i$$

a set of indexes such that the corresponding derivative of

$$u^k(X)$$

equals

$$u^k_{\beta,\Gamma}(X^\Gamma)$$

at the point X with zeros at

$$\Gamma_1, \ldots, \Gamma_n$$

places.
For example, if zeros stand at the places with the numbers l_1, l_2, l_3, then $\Gamma = (0, \ldots, 0, l_1, 0, \ldots, 0, l_2, 0, \ldots, 0, l_3, 0, \ldots, 0)$, or simply $\Gamma = (l_1, l_2, l_3)$

For example for a function u^5

$$u^5_{(2,2),(1,2)}(X^{1,2}) = \frac{\partial^4}{\partial^2 x_1 \partial^2 x_2} u^5(0, 0, x_3, \dots x_n). \qquad (2)$$

All functions

$$f_j, \; u^k_{\beta,\Gamma}(X^\Gamma)$$

are of exponential type.

LC of PDE system

Let **LC** : $u^k \mapsto U^k$.
Then

$$\mathbf{LC} : \frac{\partial^m}{\partial^{m_1} x_1 \dots \partial^{m_n} x_n} u_k(X) \mapsto (-1)^\gamma \sum_{j_1=0,\dots,j_n=0}^{m_1,\dots,m_n} p_1^{m_1-j_1} \dots p_n^{m_n-j_n} U^k_{\beta,\Gamma}(P^\Gamma)$$

Here $\gamma = \|\Gamma\|$ – the length of Γ, $\beta = (j_1, \dots, j_l)$. Note, that $U^5_{(0,0)(0,0)}(P) = U^5(P)$.

For example, in (2) for $n = 4$ we have:

LC : $\frac{\partial^4}{\partial^2 x_1 \ldots \partial^{m2} x_2} u_5(X) \mapsto$

$$p_1^2 p_2^2 U^5_{(0,0)(0,0)}(x_1, x_2, x_3, x_4)-$$
$$-p_1^2 p_2^2 U^5_{(0,0)(1,0)}(x_2, x_3, x_4) - p_1 p_2^2 U^5_{(1,0)(1,0)}(x_2, x_3, x_4)-$$

$$-p_1^2 p_2^2 U^5_{(0,0)(0,1)}(x_1, x_3, x_4) - p_1^2 p_2 U^5_{(0,1)(0,1)}(x_1, x_3, x_4)+$$

$$p_1^2 p_2^2 U^5_{(0,0)(1,2)}(x_3, x_4) + p_1 p_2^2 U^5_{(1,0)(1,2)}(x_3, x_4)+$$

$$+p_1^2 p_2 U^5_{(0,1)(1,2)}(x_3, x_4) + p_1 p_2 U^5_{(1,1)(1,2)}(x_3, x_4).$$

Denote

$$\Phi_m^k = (-1)^\gamma \sum_{j_1=0,\dots,j_l=0}^{m_1,\dots,m_n} p_1^{m_1-j_1}\dots p_n^{m_n-j_n} U_{\beta,\Gamma}^k(P^\Gamma) - P^m U^k(P)$$

$P^m = p_1^{m_1}\dots p_n^{m_n}$

As a result of Laplace-Carson transform of the system (1) according to initial condition we obtain an algebraic system relative to U^k.

$$\sum_{k=1}^{K}\sum_{m=0}^{M} a^j_{mk} U^k = F_j - \sum_{m=0}^{M} a^j_{mk}\Phi^k_m, \quad j = 1,\dots K, \tag{3}$$

Denote by D the determinant of the system (3), D_i – the maximal order minors of the extended matrix of (3). A case when there is a set S of zeros of D with infinite limit point at $Rep_k > 0, k = 1, \ldots, n$ is of most interest. Solving the system (3) we obtain U^k as fractions with D in the denominators. The inverse Laplace-Carson transform is possible if $\alpha_k, k = 1, \ldots, n$ exist such that these functions are holomorphic in the domain $Rep_k > \alpha_k$,. So we make a demand: $D_i = 0$ at S. This demand produces requirements to initial and boundary conditions, they turns to be dependent. We obtain the so-called compatibility conditions.

For example the Laplace-Carson transform of the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = f(x, y)$$

with initial and boundary conditions $u(0, y) = a(y)$, $u(x, 0) = b(x)$ is the algebraic equation

$$(p - q)U(p, q) = F(p, q) + pA(q) - qB(p).$$

Here $a(y) \mapsto A(q)$, $b(x) \mapsto B(p)$, $f(x, y) \mapsto F(p, q)$. We demand: if $p = q$, then $F(p, q) + pA(q) - qB(p) = 0$, i.e.

$$F(p, p) + pA(p) - pB(p) = 0. \tag{3}$$

The inverse Laplace-Carson transform produces the compatibility condition:

$$a(x) - b(x) + \int_o^x f(x - s, s)ds = 0.$$

The algorithm of solving the system (1) consists of three main steps:

Example

$$\begin{aligned} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} &= x, \\ \frac{\partial f}{\partial y} + \frac{\partial g}{\partial x} &= y, \end{aligned}$$

$f = f(x, y); \quad g = g(x, y)$

Initial conditions

$f(0, y) = a(y); \quad f(x, 0) = b(x); \quad g(0, y) = c(y); \quad g(x, 0) = d(x).$

$\mathbf{LC} : f(x, y) \mapsto u(p, q), \quad g(x, y) \mapsto v(p, q)$

$a(y) \mapsto \alpha(q), \quad b(x) \mapsto \beta(p)$

$c(y) \mapsto \delta(q), \quad d(x) \mapsto \gamma(p).$

$$
\begin{aligned}
pu - p\alpha(q) + qv - q\gamma(p) &= \frac{1}{p}, \\
qu - q\beta(p) + pv - p\delta(q) &= \frac{1}{q},
\end{aligned}
$$

Then

$$u = -\frac{-\alpha p^2 + \beta q^2 + (\delta - \gamma)pq}{p^2 - q^2}$$

$$v = -\frac{-p^2 + q^2 + (\alpha - \beta)p^2q^2 - (\delta p^2 - \gamma q^2)pq}{pq(p^2 - q^2)}$$

$$q = p$$

$$\alpha - \beta + \gamma - \delta = 0$$

$$\beta = 0; \quad \gamma = \frac{2}{p}; \quad \delta = \frac{2}{q}; \quad \alpha = 0;$$

$$u = -\frac{2}{p+q}$$

$$v = -\frac{p + 2p^2 + q + 2q^2 + 2pq}{pq(p+q)}$$

LC^{-1} :

$$f = -\begin{cases} 2y & , \quad y < x, \\ 2x & , \quad y \geq x, \end{cases}$$

$$g = \begin{cases} (2+y)x & , \quad y < x, \\ y(2+x) & , \quad y \geq x. \end{cases}$$

I. Laplace-Carson transform of the system (1).
II. Solving of the algebraic system (2).
III. Establishing of compatibility conditions.
IV. Inverse Laplace-Carson transform of the solutions of (2) – it is the solution of the system (1).

To provide the symbolic character of all computations we carry out the following:
1) Represent all given functions as sums (or series) of exponents with polynomial coefficients.
2) Factorize D (as full as possible).
3) Represent the solution of algebraic system as sums (or series) of algebraic fractions with exponential coefficients.

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