# New methods in potential theory

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For arbitrary homogeneous rotationally symmetric gravitating bodies, analytical methods are developed for determining their equigravitating analogues in the form of one-dimensional material line segments. The latter may have both real and imaginary density distributions, but, in any case, they preserve the total mass and generate the same extenal Newtonian force fields as the original three-dimensional bodies. For bodies with singular points (on the surface or inside), such segments may form combinations, the so-called equigravitating "skeletons." A number of examples demonstrate the usefulness of the methods. Such line segments significantly simplify the representation of external fields of self-gravitating (or electrically charged) bodies, which is important for numerous practical applications of potential theory in astronomy, celestial mechanics, and geophysics. Additionally, line segments provide a new highly effective technique for calculating the potential energy of bodies and make it possible to evaluate the radii of convergence of Laplace series independently of other known methods.

## 1. INTRODUCTION

Newton established: the external gravitational field of a sphere is equivalent to the field of a central point mass. However, the sphere is only a particular case of a spheroid.

The origin of the problem considered here was the well-known Maclaurin theorem:

*The external potentials of two homogeneous confocal ellipsoids relate to each other as their masses.*

In the limiting variants from the Maclaurin theorem it follows:

**1.** A homogeneous ellipsoid of density $\rho$ with the surface

$$\frac{x_1^2}{a_1^2}+\frac{x_2^2}{a_2^2}+\frac{x_3^2}{a_3^2}=1, \quad a_1 \ge a_2 \ge a_3,$$

has the same external potential as (or, for brevity, is equigravitating for) an *equal-mass elliptic disk*

$$\frac{x_1^2}{a_1^2-a_3^2}+\frac{x_2^2}{a_2^2-a_3^2}=1, \qquad x_3=0$$

with a surface density

$$\sigma(x_1,x_2)=\frac{2a_1a_2a_3\rho}{\sqrt{(a_1^2-a_3^2)(a_2^2-a_3^2)}}\sqrt{1-\frac{x_1^2}{a_1^2-a_3^2}-\frac{x_2^2}{a_2^2-a_3^2}}\,. \tag{1.1}$$

In special case, for an oblate spheroid $(a_1 = a_2 > a_3)$, form. (1.1) describes a round disk of radius $R=\sqrt{a_1^2-a_3^2}$ and density

$$\sigma(r)=\frac{3M}{2\pi(a_1^2-a_3^2)}\sqrt{1-\frac{r^2}{a_1^2-a_3^2}}\,, \tag{1.2}$$

which vanishes on the boundary.

**2.** A homogeneous prolate spheroid $(a_3 \ge a_1 = a_2)$ extended along the $Ox_3$ axis has the same external potential as a one-dimensional focal segment on the $Ox_3$ axis. The length of the segment is

$L=2\sqrt{a_3^2-a_1^2}$ , and it has a real linear density distribution

$$\mu(\zeta)=\frac{3M}{4\sqrt{a_3^2-a_1^2}}\left(1-\frac{\zeta^2}{a_3^2-a_1^2}\right),\ -\sqrt{a_3^2-a_1^2}\le\zeta\le\sqrt{a_3^2-a_1^2}\ . \qquad (1.3)$$

In particular, the radius of convergence of a Laplace series is equal to $L/2$ in this case.

Thus, ***an ellipsoid is represented by an equigravitating disk, and a prolate spheroid is represented by a one-dimensional line segment. However, there are no other examples provided by the Maclaurin theorem.***

The determination of equigravitating bodies is a very important and difficult problem. The greatest mathematical difficulties are associated with the calculation of the potentials of various bodies. Even for homogeneous axisymmetric bodies, the direct calculation of external potentials and especially the gravitational (potential) energy reduces to complicated two-dimensional (or five-dimensional) integrals. The calculation of the attraction between two or several bodies is another difficult problem. This motivates the development of new methods for determining equigravitating configurations that have a smaller number of spatial dimensions and are other than the examples provided by the Maclaurin theorem.

The aim of this work is to develop general analytical methods for replacing three-dimensional gravitating axisymmetric bodies by one-dimensional sources of attraction. We extend the scope of the problem and admit both real and imaginary density distributions along line segments.

## 2. CHANGING FROM REAL TO IMAGINARY LINE SEGMENTS: THE CASE OF OBLATE SPHEROIDS

As we have seen, a homogeneous oblate spheroid is modeled by a round disk with real surface density given by (1.2). There is a nontrivial generalization of this result.

**Theorem 1.** The external potential of a homogeneous oblate spheroid can be represented by the potential of a one-dimensional gravitating line segment with the imaginary density distribution

$$\mu(\zeta)=\frac{3M}{4i\sqrt{a_1^2-a_3^2}}\left(1+\frac{\zeta^2}{a_1^2-a_3^2}\right),\ -\sqrt{a_1^2-a_3^2}\le\frac{\zeta}{i}\le\sqrt{a_1^2-a_3^2}\ , \qquad (2.1)$$

*where $\zeta$ , is a purely imaginary variable ranging within the specified limits.*

**Proof.** Let us perform an analytical extension by moving from a prolate spheroid $(a_3>a_1)$ through a sphere $(a_3=a_1)$ to an oblate spheroid $(a_3<a_1)$. It is clear that hold $\sqrt{a_3^2-a_1^2}\to i\sqrt{a_1^2-a_3^2}$ so that we go directly from Eq. (1.3) to Eq. (2.1).

Verification Eq. (2.1) with respect to mass is simple:

$$M=\int\mu(\zeta)d\zeta=\frac{3M}{2i\sqrt{a_1^2-a_3^2}}\int_0^{i\sqrt{a_1^2-a_3^2}}\left(1+\frac{\zeta^2}{a_1^2-a_3^2}\right)d\zeta=\frac{4}{3}\pi a_1^2a_3\rho\,.$$

For the external potential, it is sufficient to check Eq. (2.1) only on the figure's axis of symmetry. It is well know that the classic expression for the external potential on the axis of a prolate spheroid has the form

$$\varphi_{c\phi}(x_3)=\frac{3GM}{2\sqrt{a_1^2-a_3^2}}\left\{-\frac{x_3}{\sqrt{a_1^2-a_3^2}}+\left(1+\frac{x_3^2}{a_1^2-a_3^2}\right)arctg\frac{\sqrt{a_1^2-a_3^2}}{x_3}\right\}. \qquad (2.2)$$

On the other hand, the potential of an imaginary segment on the $Ox_3$ axis is

$$\varphi_{seg}(x_3) = G\int \frac{\mu(\zeta)}{x_3 - \zeta} d\zeta$$

taking into account Eq. (2.1), this can be written as

$$\varphi_{seg}(x_3) = -\frac{3MG}{4i\sqrt{a_1^2 - a_3^2}} \int_{-i\sqrt{a_1^2-a_3^2}}^{i\sqrt{a_1^2-a_3^2}} \left(1 + \frac{\zeta^2}{a_1^2 - a_3^2}\right) \frac{d\zeta}{\zeta - x_3}. \tag{2.3}$$

It is easy to see that the change of variable $\zeta = is$ brings the integral (2.3) to the same form as (2.2). Since the external potential is a harmonic function, we can apply the well-known theorems of analysis on the uniqueness of the representations of such functions. By these theorems, since the potentials of the spheroid and the imaginary segment are equal on the $Ox_3$ axis, we conclude that they are equal at all other points of space. Thus, the homogeneous oblate spheroid has the same external potential as the imaginary segment found.

## 3. THE DIFFERENTIATION METHOD AND EQUIGRAVITATING LINE SEGMENTS FOR SHELLS

Let us consider a family of coaxial spheroidal surfaces $S(m)$

$$\frac{r^2}{a_1^2} + \frac{x_3^2}{a_3^2\alpha_3^2(m)} = m^2 \tag{3.1}$$

where $m$ is a parameter ranging within $0 \le m_{\min} \le m \le 1$. Equation (3.1) specifies the stratification of a spheroid with boundary semiaxes $a_1 \le a_3$. Two surfaces of the family separated by an infinitesimal quantity $dm$ form a spheroidal shell. The geometry of the shell is characterized by the function $\alpha_3(m)$.

Determining $\alpha_3(m)$ by the method described in [1], we can find different types of shells. Here, we dwell only on homoeoidal and confocal shells. First, let us consider the problem of determining equigravitating line segments for homogeneous homoeoids. For homothetic shells, $\alpha_3(m) = 1$, and the mass of a shell with semi-axes $a_1m$, $a_1m$ and $a_3m$ is equal to

$$M_{\text{hom}} = 4\pi a_1^2 a_3 \rho m^2 dm \tag{3.2}$$

**Theorem 2.** *A prolate thin homoeoid of mass* $\mathrm{M}_{\text{hom}}$ *with semiaxes* $a_1m \le a_3m$ *has the same external gravitational potential as a homogeneous focal segment of length* $L = 2ma_3e$ *with the constant real density distribution*

$$d_m\mu = \frac{3}{2}\frac{M}{a_3e} mdm = 2\pi\rho\frac{a_1^2}{e} mdm \tag{3.3}$$

**Proof.** The substitution of the semiaxes of the intermediate spheroid $S(m)$ into Eq. (1.3) gives

$$\mu(m,\zeta) = \frac{3M}{4a_3e}\left(m^2 - \frac{\zeta^2}{a_3^2 - a_1^2}\right), \quad e = \sqrt{1 - \frac{a_1^2}{a_3^2}} \tag{3.4}$$

Differentiating Eq. (3.4) with respect to $m$, we find the required contribution (3.3) made by the homoeoid to the density (1.3) of the entire segment.

Integration of Eq. (3.3) over the entire segment

$$\int_{-ma_3e}^{ma_3e}[d_m\mu]d\zeta$$

gives the mass of the simulated shell. To confirm that the original *homoeoid* has the same external potential as the segment, it is

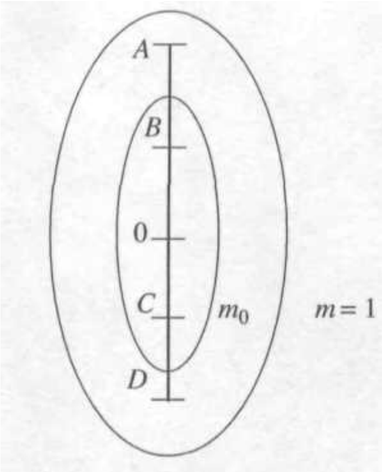

Fig. 1.

sufficient again to check this relation only at the points on the symmetry axis $Ox_3$. The potential of the segment at the point $(0, x_3)$ is

$$\varphi(x_3) = G\int \frac{[d_m\mu]}{x_3 - \zeta}d\zeta = \frac{GM_{гом}}{L}\ln\frac{x_3 + L/2}{x_3 - L/2}$$

which coincides with the external potential of the homoeoid.

Thus, an *elementary gravitating homoeoidal layer* on a prolate spheroid with semiaxes $a_1m$ and $a_3m$ and with mass (3.2) is modeled by a homogeneous real segment of the specified type. The proof is completed.

**Corollary 1.** Consider a *prolate spheroidal homoeoid* of a finite thickness (Fig 1.), whose external surface $S(m)$ with $m=1$ has focuses at $A$ and $D$ and whose internal surface with $m = m_0 < 1$ has focuses at $B$ and $C$. It is clear that $AD = 2a_3e$, and $BC = 2m_0a_3e$. The equigravitating segment for such a shell filled with a homogeneous material of density $\rho$ has the total length $AD$ and is composed of three parts. If the variable coordinate $\zeta$, along the segment falls within $AB$ or $CD$, then integrating with respect to $m$ along any of these segments gives the density

$$\mu^{\mathrm{I}}(\zeta) = \int_{\frac{\zeta}{a_3e}}^{1} d_m\mu = \pi\rho\frac{a_1^2}{e}\left(1 - \frac{\zeta^2}{a_3^2 - a_1^2}\right). \tag{3.5}$$

On $BC$, the density is uniform and is equal to

$$\mu^{\mathrm{II}}(\zeta) = 2\pi\rho\frac{a_1^2}{e}\int_{m_0}^{1} mdm = \pi\rho\frac{a_1^2}{e}\left(1 - m_0^2\right) = const \tag{3.6}$$

The verification of (3.5) and (3.6) with respect to mass is elementary:

$$M_{tot} = \frac{4}{3}\pi\rho a_1^2 a_3\left(1 - m_0^3\right) = \int_{-a_3e}^{-m_0a_3e}\mu^{\mathrm{I}}d\zeta + \int_{m_0a_3e}^{a_3e}\mu^{\mathrm{I}}d\zeta + \int_{-m_0a_3e}^{m_0a_3e}\mu^{\mathrm{II}}d\zeta$$

Verification of (3.5) and (3.6) with respect to potential is also simple, so it is omitted here.

**Corollary 2.** The results obtained for *prolate spheroidal homothetic shells* can easily be extended to oblate shells with $a_1 = a_2 > a_3$ by using the analytic continuation method. The main difference for oblate shells is that the value $L$ and the linear density distribution of the corresponding segments are purely imaginary. For example, instead of Eq. (3.3), we now have a segment of "length" $L = 2im\sqrt{a_1^2 - a_3^2}$ (for brevity, by the "length" $L$, we understand the difference between the complex numbers at the endpoints of the segment) and density

$$d_m\mu = -2\pi i\rho\frac{a_1^2 a_3}{\sqrt{a_1^2 - a_1^2}}mdm$$

Instead of (3.5) and (3.6), we have

$$\mu^{\mathrm{I}}(\zeta) = -\pi i \rho \frac{a_1^2 a_3}{\sqrt{a_1^2 - a_3^2}} \left(1 + \frac{\zeta^2}{a_3^2 - a_1^2}\right), \quad \mu^{\mathrm{II}}(\zeta) = -\pi i \rho \frac{a_1^2 a_3}{\sqrt{a_1^2 - a_3^2}} \left(1 - m_0^2\right) = const.$$

respectively.

Let us determine equigravitating *line segments for focaloids*. For a confocal stratification of a prolate spheroid, the semiaxes of an intermediate surface in (3.2) are [1]

$$a_1(m) = a_3 \sqrt{m^2 - e^2}, \quad a_3(m) = a_3 m. \qquad (3.7)$$

**Theorem 3.** *A prolate thin spheroidal focaloid bounded by surfaces* $S(m)$ *and* $S(m + dm)$ *with the mass*

$$M_{foc}(m) = \frac{4}{3} \pi \rho a_3^3 (3m^2 - e^2) dm$$

*has the same external gravitational potential as a line segment of length* $L = 2ea_3$ *with the real density distribution*

$$d_m \mu(m, \zeta) = \pi \rho a_3^2 \frac{3m^2 - e^2}{e} \left(1 - \frac{\zeta^2}{a_3^2 e^2}\right) dm, \quad -ea_3 \le \zeta \le ea_3. \qquad (3.8)$$

**Proof.** We again prove the theorem by the differentiation method. The substitution of (3.7) for $a_i$ in Eq. (1.3) gives

$$\mu(m, \zeta) = \pi \rho a_3^2 \frac{m(m^2 - e^2)}{e} \left(1 - \frac{\zeta^2}{e^2 a_3^2}\right) \qquad (3.9)$$

Differentiating Eq. (3.9) with respect to $m$, we can easily find that the contribution from an elementary gravitating focaloid of the specified type is expressed by (3.8).

**Corollary 3.** *A focaloid of finite thickness* bounded by the surfaces $S(1)$ and $S(m_0)$ has the same external potential as a line segment with the density distribution

$$\mu(m_0, \zeta) = \frac{\pi \rho}{e} \left(1 - m_0\right) \left(1 - e^2 + m_0 + m_0^2\right) \left(a_3^2 - \frac{\zeta^2}{e^2}\right) \qquad (3.10)$$

This relationship is obtained by integrating Eq. (3.8) with respect to $m$ from $m_0$ to 1. In particular, when $m_0 = m_{\min}^2 = e^2(1) = 1 - a_1^2 / a_3^2$, the focaloid transforms into a continuous spheroid, with the internal cavity of the shell degenerating into an infinitely thin geometric slit in the form of a focal segment. In this case, we go back from Eq. (3.10) to Eq. (1.3).

**Corollary 4.** For stratification into *oblate focaloids*, we obtain, instead of (3.7), the semiaxes [3]

$$a_1 m, \ a_1 m, \ a_1 \sqrt{m^2 - e^2}, \quad e = \sqrt{1 - a_3^2 / a_1^2},$$

and Eq. (3.9) is replaced by

$$\mu(m, \zeta) = -\frac{i \pi \rho}{e} m^2 \sqrt{m^2 - e^2} \left(a_1^2 + \frac{\zeta^2}{e^2}\right)$$

Using the same method as above, we obtain the density distributions of the line segments for a *thin oblate focaloid and an oblate focaloid of finite thickness* (cf. Eqs. (3.8) and (3.10))

$$d_m\mu(m,\zeta) = -i\pi\rho\frac{m(3m^2-2e^2)}{e\sqrt{m^2-e^2}}\left(a_1^2+\frac{\zeta^2}{e^2}\right),$$

$$d_m\mu(m_0,\zeta) = -\frac{i\pi\rho}{e}\left(\frac{a_3}{a_1}-m_0\sqrt{m_0^2-e^2}\right)\left(a_1^2+\frac{\zeta^2}{e^2}\right), \quad -a_1e\le\frac{\zeta}{i}\le a_1e$$

**Remark 1.** The formulas for line segments associated with focaloids can easily be verified with respect to mass and potential, so this verification is omitted here.

We emphasize that the above examples are not based on the Maclaurin theorem. Their significance will be seen from the following analysis.

## 4. REPRESENTATION OF THE POTENTIAL OF A FLAT DISK BY AN IMAGINARY GRAVITATING LINE SEGMENT

Suppose that a *homogeneous flat disk* of radius $R$ with surface density $\sigma$ is given (Fig. 2a).

**Theorem 4.** *The potential of a disk of the specified type at an external point* $(r, x_3)$ *is represented by the potential of a line segment of length* $L = 2iR$ *with the imaginary linear density*

$$\mu(\zeta) = -2i\sigma\sqrt{R^2+\zeta^2}\ ,\ -R\le\frac{\zeta}{i}\le R\,. \tag{4.1}$$

**Proof.** Let us project the mass of the disk onto the diameter lying on the axis $Ox_1$. This gives a one-dimensional real segment of length $2R$ with the mass $dM = \mu(x_1)dx_1$ in the range between $x_1$ and $x_1+dx_1$ and with the linear density

$$\mu(x_1) = 2\sigma\sqrt{R^2-x_1^2} \tag{4.2}$$

We shift in (4.2) to the variable

$$\zeta = ix_1, \tag{4.3}$$

which assigns purely imaginary values to all segment points, and take into account $dM = \mu(\zeta)d\zeta$ obtain the imaginary density law (4.1). The length of the segment also becomes purely imaginary: $L = 2iR$. The change of variable (4.3) refers only to the segment points and rotates the segment by 90° from the plane $Ox_1x_2$ (Fig. 2b).

The mass of the imaginary segment,

$$M = \int_{-ia}^{ia}\mu(\zeta)d\zeta = 4\sigma\int_0^a\sqrt{R^2-s^2}\,ds = \pi\sigma R^2$$

is real and is equal to the mass of the original disk. By definition, the potential of the segment on the axis of symmetry at $(0, x_3)$ is given by the integral

$$\varphi_{seg}(x_3) = -2i\sigma G\int_{-ia}^{ia}\frac{\sqrt{R^2+\zeta^2}}{x_3-\zeta}d\zeta$$

Making the substitution $\zeta = is$ and getting rid of the imaginary denominator, after evident transformations, we find

$$\varphi_{seg}(x_3) = 2\sigma G\,|\,x_3\,|\int_{-a}^{+a}\frac{\sqrt{R^2-s^2}}{x_3^2+s^2}ds = 2\pi G\sigma\left(\sqrt{R^2+x_3^2}-|\,x_3\,|\right) \tag{4.4}$$

On the other hand, it is easy to see that the direct calculation of the disk potential

$$\varphi_d(x_3) = 2\pi G\sigma\int_0^R\frac{rdr}{\sqrt{r^2+x_3^2}}$$

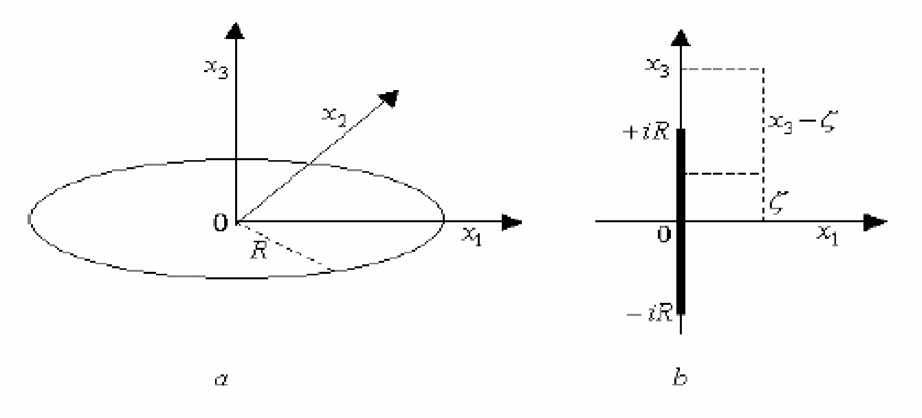


Fig.2

yields the same result. Since the external potential is a harmonic function, in accordance with well-known theorems of the analysis, the above equality between the potentials of the disk and the segment on the axis $Ox_3$ ensures that they are equal at any other point of space. The proof is completed.

**Corollary 5.** A one-dimensional ring of radius $a$ with a uniform linear density $\mu_0$ has the same potential as an imaginary line segment with infinite densities at the endpoints:

$$\mu(\zeta) = -\frac{2iR\mu_0}{\sqrt{R^2+\zeta^2}}, \ -R \le \frac{\zeta}{i} \le R. \tag{4.5}$$

**Proof.** Differentiating (4.1) with respect to and taking into account $\mu_0 = \sigma\, dR$, we readily obtain Eq. (4.5). The potential of the segment at $(r, x_3)$ is

$$\varphi(r,x_3) = G\int_{-iR}^{iR} \frac{\mu(\zeta)d\zeta}{\sqrt{r^2+(x_3-\zeta)^2}} = \frac{4GR\mu_0}{\sqrt{(R+r)^2+x_3^2}} K\left(\sqrt{\frac{4Rr}{(R+r)^2+x_3^2}}\right),$$

where $K$ is a complete elliptic integral of the first kind. It is easy to see that this potential is equivalent to the ring's potential calculated directly.

## 5. DETERMINATION OF EQUIGRAVITATING LINE SEGMENTS FOR THREE-DIMENSIONAL BODIES BY DIVISION INTO DISKS

From disks (and rings), we can construct axisymmetric bodies of various forms. A synthesis method (basing on equigravitating line segments for disks and rings) can be employed to determine equigravitating line segments for such total bodies. Let there be a homogeneous axisymmetric body (Fig. 3a), whose meridional cross section is described by a function $a = a(h)$, where a is the radius of the disk at the height $h$. It is assumed that the surface and the interior of the body may have a finite number of singular points (The question of the existence and the number of singular points requires the investigation of the complex variable functions $\mu(\zeta)$ from Eq. (5.1) or, equivalently, of the potential $\varphi(\zeta)$). These singular points serve as endpoints of one-dimensional material line segments (one or several, see justification in Section 6). For such segments to be equigravitating for a three-dimensional body, the following theorem must be valid

**Theorem 5.** *For a homogeneous axisymmetric body, the density distribution of one-dimensional gravitating segments at point* $\zeta$ *, is given by the integral*

$$\mu(\zeta) = -2i\rho\int_{h_1}^{h_2} \sqrt{a^2(h)+(\zeta-h)^2}\,dh \tag{5.1}$$

**Proof.** Suppose that the body consists of round flat disks of length $dh$ with surface density $\rho dh$. Then, according to Eq. (4.1), the potential of any of the intermediate disks can be represented by an elementary gravitating segment (Fig. 3b) with the density distribution

$$d\mu(\zeta) = -2i\rho\sqrt{a^2(h) + (\zeta - h)^2}\, dh \tag{5.2}$$

and the potential itself is expressed by

$$d\varphi(x_3) = -2iG\rho dh \int \frac{\sqrt{a^2(h) + (\zeta - h)^2}}{x_3 - \zeta} d\zeta$$

The total potential is found by integrating the contributions from all elementary segments

$$\varphi(x_3) = -2iG\rho \iint \frac{\sqrt{a^2(h) + (\zeta - h)^2}}{x_3 - \zeta} dh d\zeta \tag{5.3}$$

Any integral over an elementary segment is replaced by a half of the integral along the contour tightly encircling that segment (Fig. 3b). Extending this contour up to a meridional cross-section of the body (which is possible, because we will not find any singular point when extending the contour) and doing so with the

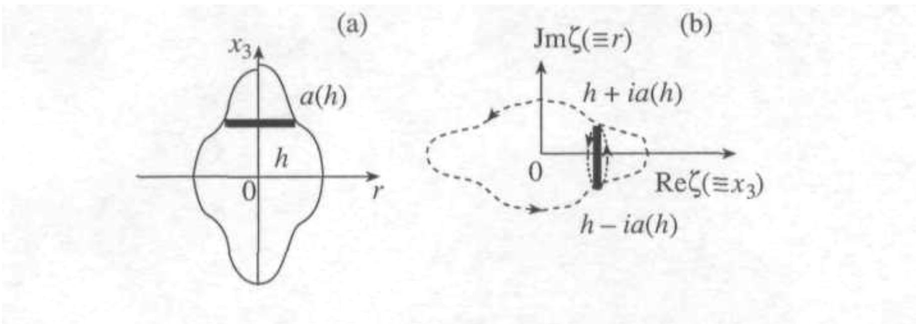


**Fig. 3.**

contours enclosing all other elementary segments, we replace the potential (5.3) of the entire body by the contour integral

$$\varphi(x_3) = -iG\rho \oint_C \frac{d\zeta}{x_3 - \zeta} \left( \int_{h_1}^{h_2} \sqrt{a^2(h) + (\zeta - h)^2}\, dh \right) \tag{5.4}$$

It should be emphasized that changing from (5.3) to (5.4) is possible, because the integration over the intermediate segments was replaced by the integration along the same contour $C$, which is the section of the body by the complex plane $\zeta = x_3 + ir$. However, as known, the sum of such contour integrals can be replaced by a single contour integral of the sum of the integrands. Since the contour integral (5.4) is reduced, in turn, to one or several integrals over one-dimensional line segments (for more detail on reduction, see Subsection 6.2), the total density on each final segment is given by (5.1). The proof is completed.

**Remark 2.** The following should be said regarding the choice of the limits of integration in Eq. (5.1). If the order of the algebraic equation

$$a^2(h) + (\zeta - h)^2 = 0 \tag{5.5}$$

for the unknown $h$ is greater than or equal to two, then both limits of integration $h_1$ and $h_2$ must be among the roots of this equation. If Eq. (5.5) is of the first order, it has only one solution $h_2$, and the lower limit for $h$ should be chosen from additional conditions of the problem.

**Example 1.** Use this method to find an equigravitating line segment for an *oblate spheroid*. In this case,

$$a^2(h) = a_1^2\left(1 - h^2/a_3^2\right),$$

and Eq. (5.1) may be written as

$$\mu(\zeta) = -2i\rho \int_{h_1}^{h_2} \sqrt{-\left(\frac{a_1^2}{a_3^2} - 1\right)h^2 - 2\zeta\, h + \zeta^2 + a_1^2}\, dh \tag{5.6}$$

Here, $h_1$ and $h_2$ are the roots of the quadratic equation

$$h^2 + \frac{2a_3^2\zeta}{a_1^2 - a_3^2} h - \frac{a_3^2(\zeta^2 + a_1^2)}{a_1^2 - a_3^2} = 0\,,$$

They are

$$h_{1,2} = \frac{a_1 a_3\left(-\zeta \dfrac{a_3}{a_1} \pm \sqrt{\zeta^2 + a_1^2 - a_3^2}\right)}{a_1^2 - a_3^2}$$

Here, there are two singular points, so that the line segment is unique. Therefore, the integration of Eq. (5.6), represented as

$$\mu(\zeta) = -2i\rho \frac{\sqrt{a_1^2 - a_3^2}}{a_3} \int_{h_1}^{h_2} \sqrt{(h_2 - h)(h - h_1)}\, dh$$

yields

$$\mu(\zeta) = -2i\rho \frac{\sqrt{a_1^2 - a_3^2}}{a_3}\left[\frac{\pi}{8}(h_2 - h_1)^2\right] = -i\pi\rho \frac{a_1^2 a_3}{\sqrt{a_1^2 - a_3^2}}\left(1 + \frac{\zeta^2}{a_1^2 - a_3^2}\right)$$

This result for a spheroid coincides with the distribution (2.1) obtained earlier by another method.

**Example 2.** *The replacing rod for a spherical segment*. Cutting the latter by a plane at the height $h$ (Fig. 4), we obtain a round disk of radius

$$a(h) = \sqrt{b^2 - 2h(R - H) - h^2}$$

and of thickness $dh$. Then, for the spherical segment, according to Eqs. (5.2) and (5.1), we have

$$\mu(\zeta) = -2i\rho \int_{h_1}^{h_2} \sqrt{b^2 + \zeta^2 - 2h(\zeta + R - H)}\, dh \tag{5.7}$$

The integrand in Eq. (5.7) is an expression of the first order in $h$. It follows that

$$h_2(\zeta) = \frac{b^2 + \zeta^2}{\zeta + R - H}$$

It is easy to see that $h_1 = 0$ in this example. With the specified limits, the integral (5.7) is equal to

$$\mu(\zeta) = -\frac{2}{3} i\rho \frac{\left(b^2 + \zeta^2\right)^{3/2}}{\zeta + R - H}\,, \quad -b \le \zeta/i \le b\,. \tag{5.8}$$

The verification of this formula by integrating with respect to $\xi$ immediately confirms that the mass of the line segment

$$M = \pi\rho H^2 \left(R - H/3\right)$$

is equal to the mass of the spherical segment. It can be shown [4] that the potential of the line segment with density (5.8) at point $(0, x_3)$ is

$$\varphi(x_3) = \frac{2}{3}\pi G\rho\left[-\frac{R^3 + (b^2 + x_3^2)^{3/2}}{H - R - x_3} - x_3^2 + (R - H)x_3 - R^2 - RH + \frac{H^2}{2}\right],$$

which agrees with the external potential of the spherical segment calculated by the direct method.

## 6. DETERMINATION OF EQUIGRAVITATING LINE SEGMENTS FOR AXISYMMETRIC BODIES WITH THE HELP OF THE MODIFIED CAUCHY INTEGRAL

Application of the theory of complex variable to the problem makes it possible to do the following:

(a) To prove the existence of equigravitating segments for bodies with azimuthal symmetry;

(b) To find imaginary and real linear density distributions of mass on these segments by an independent method.

### *6.1. Extension of the Cauchy Integral to the Newtonian Potential*

Since an axisymmetric body is in fact two-dimensional and only two coordinates $x_3$ and $r = \sqrt{x_1^2 + x_2^2}$ are sufficient to describe its geometry, we can introduce the complex plane $\zeta = x_3 + ir$. On this plane, a body *T* is encircled by two contours $\Gamma$ and $\Gamma_1$ with the tested point $P(0, x_3)$ of the polar axis lying between them

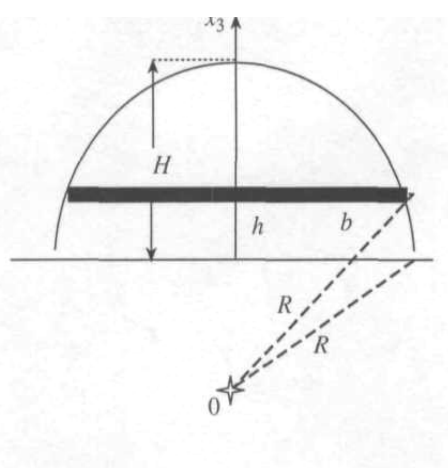

Fig. 4.

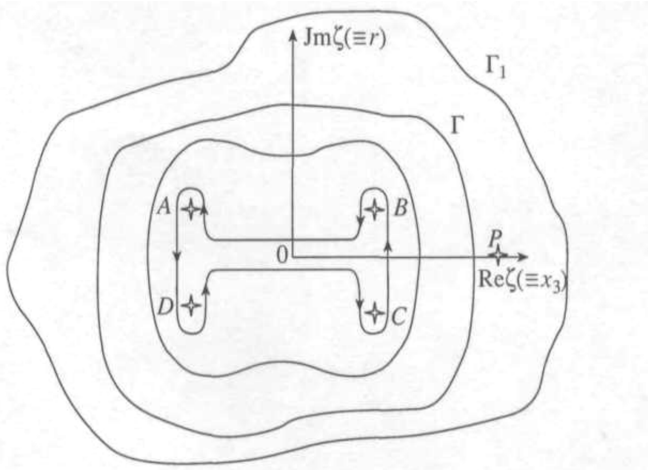

Fig. 5.

(Fig. 5). Then, according to the Cauchy integral theory, the potential of the body at $P$ can be represented as [5]

$$\varphi(x_3) = \frac{1}{2\pi i}\oint_{\Gamma_1}\frac{\varphi(\zeta)}{\zeta - x_3}d\zeta - \frac{1}{2\pi i}\oint_{\Gamma}\frac{\varphi(\zeta)}{\zeta - x_3}d\zeta\,. \tag{6.1}$$

We extend indefinitely the outer contour $\Gamma_1$. Then, because the potential of the body goes to zero at infinity, the first contour integral in Eq. (6.1) vanishes, and only a modified Cauchy integral remains:

$$\varphi(x_3) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\varphi(\zeta)}{x_3 - \zeta} d\zeta \, . \tag{6.2}$$

### *6.2. Reduction of Γ to Segments*

The remaining integral (6.2) is nonzero, because $\varphi(\zeta)$ necessarily has singular points A, B, ... inside $\tilde{A}$. The function $\varphi(\zeta)$ is obtained by substituting $\zeta = x_3 + ir$ for $x_3$ in the potential $\varphi(x_3)$ of the body on the symmetry axis at the external point $(0, x_3)$. Strictly speaking, $\varphi(x_3)$ may be multivalued, and we should choose its branch that behaves like $\varphi(\zeta) \sim \zeta^{-1}$ at very large $\zeta$. Without changing the value of integral (6.2), we contract Γ from the most distant neighborhood towards the body's inside and deform this contour so that it is spanned by the singular points. The resulting figure is a polygon (if the number of singular points is greater than two). We emphasize that the contraction of the contour is an analytical extension of the chosen branch of $\varphi(\zeta)$. Finally, we transform the contour polygon into a set of line segments forming an equigravitating framework for the original body. In the extreme case, we obtain one (if there are two singular points) or several (if there are more then two singular points) equigravitating segments (Fig. 5).

Let us consider one of the line segments obtained in this way, for example, *BC* (Fig. 6). In accordance with Eq. (6.2), by taking into account the common rule of signs for $d\zeta$, the linear density $\mu(\zeta)$ is determined by the difference between two branches of the function:

$$\mu(\zeta) = \frac{1}{2\pi i G} [\varphi_{\mathrm{I}}(\zeta) - \varphi_{\mathrm{II}}(\zeta)] \tag{6.3}$$

**Example 3.** Consider a *homogeneous flat disk* (Fig. 2a), with the potential $\varphi(x_3)$ given by (4.4). By making the substitution $x_3 \to \zeta = x_3 + ir$ in (4.4), the image of the disk rotates by about 90° (Fig. 2), and the potential takes the form

$$\varphi(\zeta) = 2\pi G\sigma(\sqrt{a^2 + \zeta^2} - \zeta)$$

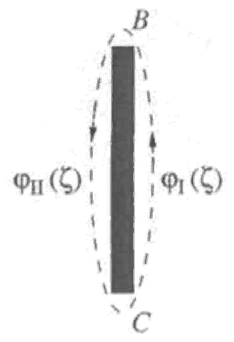


**Fig. 6.**

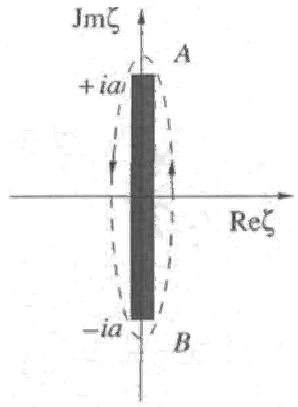


**Fig. 7.**

and has the required asymptotics $\varphi \sim \zeta^{-1}$ at (much more than) $|\zeta| >> a$. Here, the singular points are $\zeta = \pm ia$, so that Γ gives, in the limit, only one segment *BA* (Fig. 7). When traveling around the singular points *A* and *B,* the radical $\sqrt{a^2 + \zeta^2}$ is known to change its sign, so that

$$\varphi_{\mathrm{I}}(\zeta) = 2\pi G\sigma\left(\sqrt{a^2 + \zeta^2} - \zeta\right), \quad \varphi_{\mathrm{II}}(\zeta) = -2\pi G\sigma\left(\sqrt{a^2 + \zeta^2} + \zeta\right).$$

According to (6.3), the density of *BA* is equal to

$$\mu(\zeta) = -2i\sigma\sqrt{a^2+\zeta^2}\ ,\ -a \le \frac{\zeta}{i} \le a\,,$$

which agrees with Eq. (4.1) found by another technique.

**Example 4.** Find equigravitating line segments for a *homogeneous round cylinder* bounded in height (Fig. 8a).

After the substitution $x_3 \to \zeta$, the external potential of this body on the axis $Ox_3$ is represented as

$$\varphi(\zeta) = \pi G\rho\left[(\zeta+H)\sqrt{a^2+(\zeta+H)^2} - (\zeta-H)\sqrt{a^2+(\zeta-H)^2} + a^2\ln\frac{\zeta+H+\sqrt{a^2+(\zeta+H)^2}}{\zeta-H+\sqrt{a^2+(\zeta-H)^2}} - 4H\zeta\right]. \quad (6.4)$$

The last term in (6.4) can also be dropped (see the example with a disk), because it makes no contribution to the contour integral (6.2). The singular points of $\varphi(\zeta)$ are $\zeta = -H \pm ia$ and $\zeta = H \pm ia$. After reduction, $\Gamma$ reduces to three line segments (Fig. 8b). Thus, we have the segment *CB* with the endpoints at $H - ia$ and $H + ia$ and with the density

$$\mu_1(\zeta) = -\frac{1}{2}i\rho\left[-2(\zeta-H)\sqrt{a^2+(\zeta-H)^2} + a^2\ln\frac{\zeta-H-\sqrt{a^2+(\zeta-H)^2}}{\zeta-H+\sqrt{a^2+(\zeta-H)^2}}\right]$$

and the segment *DA* with the endpoints at $-H - ia$ and $-H + ia$ and with the density

$$\mu_2(\zeta) = -\frac{1}{2}i\rho\left[2(\zeta+H)\sqrt{a^2+(\zeta+H)^2} + a^2\ln\frac{\zeta+H+\sqrt{a^2+(\zeta+H)^2}}{\zeta+H-\sqrt{a^2+(\zeta+H)^2}}\right]$$

Since the values of $\ln\left[\zeta - H + \sqrt{a^2+(\zeta-H)^2}\right]$ at points 1 and 2 in Fig. 8b differ by $2\pi i$, we have the third segment *EF* of length $-H \le \zeta \le H$ with the uniform real density

$$\mu_3(\zeta) = \pi\rho a^2 = const\,.$$

It is important that only *EF* contributes to the mass of the cylinder:

$$M = 2\pi\rho a^2 H\,,$$

because the imaginary contributions from *CB* and *DA*

$$M_1 = -i\pi\rho a^3,\ \ M_2 = i\pi\rho a^3$$

cancel each other out. Thus, all three line segments found contribute to the potential.

**Example 5.** Find an equigravitating line segment for a *thin spherical cap*.

Let the cap *ACB* on a sphere of radius R have the apex angle $2\alpha$ and the surface density $\sigma$ (Fig. 9). Suppose that the origin is placed at the sphere center, and $Ox_3$ is the axis of symmetry of the cap. After the substitution $x_3 \to \zeta$, the potential of the cap at the point $(0, x_3)$ is represented by the formula

$$\varphi(\zeta) = 2\pi G\sigma R\frac{\sqrt{\zeta^2 - 2R\zeta\cos\alpha + R^2} - R + \zeta}{\zeta} \quad (6.5)$$

at (more than) $\zeta > R$. In this case, there are two singular points $\zeta = \mathrm{Re}^{\pm i\alpha}$, so that the length of the required segment is $L = 2iR\sin\alpha$. Only the radical in the numerator of (6.5) contributes to the contour integral (and, therefore, to the density of the segment). Hence, the equigravitating line segment has the density distribution

$$\mu(\zeta) = -2iR\sigma \frac{\sqrt{\zeta^2 - 2R\zeta\cos\alpha + R^2}}{\zeta}, \quad L = \mathrm{R}(\mathrm{e}^{i\alpha} - \mathrm{e}^{-i\alpha})$$

It is clear that the radius of convergence of the Laplace series also equals $L/2i$ in this case.

## 7. EQUIGRAVITATING BODIES. CONFOCAL TRANSFORMATIONS OF SHELLS AND ELLIPSOIDS

### 7.1. Confocal transformations ellipsoidal shells and stratified inhomogeneous ellipsoids

Let consider the ellipsoid with internal stratification

$$\frac{x_1^2}{\alpha_1^2 m^2} + \frac{x_2^2}{\alpha_2^2 m^2 \alpha_2^2(m)} + \frac{x_3^2}{\alpha_3^2 m^2 \alpha_3^2(m)} = 1 \quad \left(0 \le m_{\min} \le m \le 1\right). \tag{7.1}$$

The intermediate surface $S(m)$ has squares semiaxes

$$\alpha_1^2 m^2,\ \alpha_2^2 m^2 \alpha_2^2(m),\ \alpha_3^2 m^2 \alpha_3^2(m). \tag{7.2}$$

Let's execute now such transformation of this intermediate surface to another ellipsoidal a surface with *new* squares semiaxes

$$m^2\left(\alpha_1^2 + \mu\right),\quad m^2\left(\alpha_2^2\alpha_2^2(m) + \mu\right),\quad m^2\left(\alpha_3^2\alpha_3^2(m) + \mu\right). \tag{7.3}$$

At the given transformations focuses of surface $S$ will coincide with focuses of a new surface $S(m, \mu x)$. *Transformations* (7.3) *we shall name for brevity as confocal one.* It is emphasize, that here we expand concept of confocal transformations and apply them not only to homoeoids and focaloids, but also to ellipsoidal shells, and then to *stratified inhomogeneous* ellipsoids. In the classical literature confocal transformations were applied only to continuous ellipsoids and to thin homoeoids.

For separate ellipsoidal shells the important statement takes place:

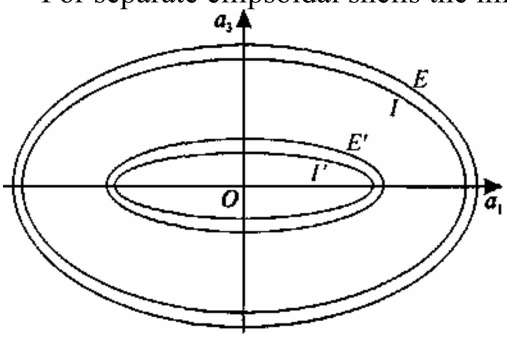


**Fig. 8.** Two ellipsoidal shells are confocal if those semiaxes are connected by transformation (7.3)

1). *When two ellipsoidal shells, thick or thin, are connected by transformation* (7.3), *their external surfaces* E *and* E′ , *as their internal* I *and* I′, are *confocal. We shall name such shells as confocal with each another.*

2). Except for it, *the important property of the transformations* (7.3) *is also that they keep geometrical type of an initial shell.* So, homoeoid with semiaxes $\alpha_i m_E\,(i = 1, 2, 3)$ for an external surface E, and $\alpha_i m_I$ for an internal surface I, transforms in homoeoid with semiaxes $\sqrt{\alpha_i^2 + \mu}$ and $m_I\sqrt{\alpha_i^2 + \mu}$. Both homoeoids will be confocal each other though have different, certainly, oblateness of the main sections, for example, $\varepsilon_{12} = 1 - \frac{\alpha_2}{\alpha_1}$ and $\varepsilon'_{12} = 1 - -\sqrt{\frac{\alpha_2^2 + \mu}{\alpha_1^2 + \mu}}$ . Further, focaloids will be transformed in such a way also in focaloids, etc.

3. However transformations (7.3) change, generally speaking, volume of shells. For *preservation of volume it is necessary to enter the additional requirement.* Namely, if two elementary confocal shells to fill with homogeneous substance on density a condition of preservation of their weight will be

$$\rho(m)\alpha_1\alpha_2\alpha_3 \frac{d}{dm}\left[m^3\alpha_2(m)\alpha_3(m)\right] =$$

$$= \rho'(m)\frac{d}{dm}\left[m^3\sqrt{\left[\alpha_1^2 + \mu\right]\left[\alpha_2^2\alpha_2^2(m) + \mu\right]\left[\alpha_3^2\alpha_3^2(m) + \mu\right]}\right], \tag{7.4}$$

Where $\rho(m)$ also $\rho'(m)$ - *density* of the initial and transformed shells.

### 7.2. Equigravitating ellipsoidal shells

*The major property of two elementary shells connected by transformations* (7.3), *that at identical mass they have in external space the same Newtonian potential, as it such confocal shells are equigravitating.*

#### 7.2.1. Confocal homoeoids

It is given elementary gravitating homoeoid with homogeneous density and weight *of the M,* having a boundary surface with semiaxes $\alpha_i$:

$$\sum_{i=1}^{3}\frac{x_i^2}{\alpha_i^2} = 1. \quad (7.5)$$

Let's consider another elementary homoeoid the same weight, having semiaxes $\sqrt{\alpha_i^2 + \lambda_0}$ and confocal to the first. A surface of this homoeoid

$$\sum_{i=1}^{3}\frac{x_i^2}{\alpha_i^2 + \lambda_0} = 1. \quad (7.6)$$

**The theorem 1 (Chasles)**. *Two elementary confocal homoeoids of the given type create identical potentials in an external point* $(\tilde{x}_i)$.

**PROOF.** Let's consider external potentials of these confocal shells. According to (5.64), potentials for the first and the second homoeoids (their masses are identical) are accordingly equal

$$\varphi_1(\tilde{x}_i) = \frac{GM}{2}\int_{\lambda_1}^{\infty}\frac{ds}{\sqrt{(\alpha_1^2+s)(\alpha_2^2+s)(\alpha_3^2+s)}} \tag{7.7}$$

and

$$\varphi_2(\tilde{x}_i) = \frac{GM}{2}\int_{\lambda_1-\lambda_0}^{\infty}\frac{ds}{\sqrt{(\alpha_1^2+\lambda_0+s)(\alpha_2^2+\lambda_0+s)(\alpha_3^2+\lambda_0+s)}}. \tag{7.8}$$

Ellipsoidal coordinates of test points are the greatest decisions of the cubic equations

$$\sum_{i=1}^{3}\frac{\tilde{x}_i^2}{\alpha_i^2+\lambda_1} = 1, \quad \sum_{i=1}^{3}\frac{\tilde{x}_i^2}{\alpha_i^2+\lambda_1-\lambda_0} = 1. \tag{7.9}$$

Then replacement

$$\lambda_0 + s = s' \tag{7.10}$$

is reduced the potential (7.8) to potential (7.7). So $\varphi_1 = \varphi_2$.

### 7.2.2. Confocal focaloids

Actually, this case has been already considered by us above.

### 7.2.3. Confocal ellipsoidal shells of the general type

Let's allocate from family (7.1) at the fixed small value of parameter $dm$ any elementary ellipsoidal shell. Then other elementary shell received from given by replacement semiaxes (7.3), appears, as we know, confocal. Masses of both shells we believe equal each other.

**The theorem 2.** *Two confocal elementary ellipsoidal homogeneous shells of the given type have identical gravitational potentials in external space.*

**PROOF.** Potentials of both shells in an external point $\tilde{x}_i$ are given by the expressions

$$d\varphi_1^0\left(\tilde{x}_i, m\right) = \frac{3}{4} G \left\{ dM(m) \int\limits_{\lambda_1\left(m^2\right)}^{\infty} F_1\left(m^2, \upsilon\right) d\upsilon + M(m) dm \int\limits_{\lambda_1\left(m^2\right)}^{\infty} \frac{d}{dm}\left[F_1\left(m^2, \upsilon\right)\right] d\upsilon \right\} \qquad (7.11)$$

$$d\varphi_2^0\left(\tilde{x}_i, m\right) = \frac{3}{4} G \left\{ dM(m) \int\limits_{\lambda_2\left(m^2\right)}^{\infty} F_2\left(m^2, \upsilon\right) d\upsilon + M(m) dm \int\limits_{\lambda_2\left(m^2\right)}^{\infty} \frac{d}{dm}\left[F_2\left(m^2, \upsilon\right)\right] d\upsilon \right\} \qquad (7.12)$$

where ellipsoidal coordinates of test points $\left(\tilde{x}_i\right)$ are the greatest roots of the cubic equations

$$\sum_{i=1}^{3} \frac{\tilde{x}_i^2}{\alpha_i^2 m^2 \alpha_i^2(m) + \lambda_1\left(m^2\right)} = 1; \quad \sum_{i=1}^{3} \frac{\tilde{x}_i^2}{\alpha_i^2 m^2 \alpha_i^2(m) + m^2\mu + \lambda_2\left(m^2\right)} = 1. \qquad (7.13)$$

Then, the replacement $\upsilon' = m^2\mu + \upsilon$ expression $F_2$ turns back in $F_1$ and the bottom limit in integral (7.12) becomes equal $\lambda_2\left(m^2\right) + m^2\mu$. But as equality takes place

$$\lambda_2\left(m^2\right) + m^2\mu = \lambda_1\left(m^2\right), \qquad (7.14)$$

That is obvious, that the potential (7.12) is reduced to initial potential (7.11.)

As external potentials are proportional to masses of bodies, fairly simple generalization.

**The theorem 3.** *External potentials elementary confocal (in sense entered above transformations* (7.3)) *homoeoids, focaloids and in general any ellipsoidal shells are proportional to masses of these bodies*

$$\frac{\varphi_1}{\mu_1} = \frac{\varphi_2}{\mu_2} = \ldots = \frac{\varphi_i}{\mu_i}. \qquad (7.15)$$

It is important to notice, that the theorem 3 extends not only on thin, but also on thick ellipsoidal shells, and also on continuous stratified inhomogeneous ellipsoids with distribution of density $\rho = \rho\left(m^2\right)$. Thus, formulating this theorem, we leave far for frameworks of classical theorem Maclaurin - Laplase and theorem Chasles 1.

### 7.3. The theorem about equigravitating stratified inhomogeneous ellipsoids

Let's consider gravitating stratified inhomogeneous ellipsoid. Its external potential is given by the formula (6.121). We shall create now from it another ellipsoid, transforming under formulas (7.3) each of intermediate elementary shells of the first in a confocal shells. Masses of the initial and transformed layers it is believed equal. Then the following theorem takes place.

**The theorem 4.** *Two stratified inhomogeneous ellipsoids the given type, layers at which are connected by confocal transformations* (7.3), *are equigravitating in an external point* $\tilde{x}_i$ .

**PROOF.** According to the theorem 3, two confocal elementary shells connected by transformation (7.3), under condition of preservation of mass are equigravitating ones. But as potential is additive function on mass, then stratified inhomogeneous ellipsoids are equigravitating ones in external space.

**REMARK 1.** Results of the theorem 4 extend and on shells of final thickness.

## 8. CONCLUSIONS

We have seen that the methods developed here (differentiation of shells, division into disks or rings, and the method based on the Cauchy integral) make it possible to find equigravitating line segments for homogeneous axisymmetric bodies. The last two methods are equivalent. The first two methods can be employed to analyze bodies with internal cavities. The above examples give greater insight into the theory developed.

An interesting law can be pointed out on the basis of the results of this study. If the potential of a body is represented by a flat disk with a *real* density distribution, then the same potential can be represented by a one-dimensional line segment with an *imaginary* density. Conversely, an *imaginary* disk corresponds to a *real* line segment.

The replacement of three-dimensional gravitating bodies by one-dimensional line segments provides an effective technique for solving many problems in potential theory and can have numerous practical applications in physics and geophysics (for example, in direct and inverse problems of gravitational prospecting), celestial mechanics (including the problem of mutual attraction of bodies), and astrophysics. In particular, the calculation of the mutual attraction of bodies is simplified significantly by using line segments. In this connection, the method for determining the potential energy of bodies based on their one-dimensional equigravitating skeletons should be developed.

For search of new equigravitating bodies we develop one more method, actively using *special confocal transformations.* Such transformations are applied at us not only to homogeneous continuous ellipsoids ( (as it made classics), but also to separate elementary ellipsoidal shells *of the general type* (and not just to homoeoids as it made Chasles), and also to *ellipsoidal thick shells* and to *stratified inhomogeneous* ellipsoids as a whole. We proved that elementary (or final thickness) ellipsoidal shells and continuous stratified inhomogeneous ellipsoids, connected by the specified confocal transformations, are *equigravitating ones.*

## REFERENCES


1. Kondratyev, B.P., *Dinamika ellipsoidal'nykh gravitiruyushchikh figur* (Dynamics of Ellipsoidal Gravitating Fig ures), Moscow: Nauka, 1989.
2. Kondratyev, B.P. The potential theory and equilibrium figures. Moscow-Izhevsk: The Institute of Computer Science, 2003.
3. Kondratyev, B.P. The potential theory. New methods and problems with solutions. Moscow: Mir, 2007.