Numerical and analytical study of impedance/frequency and parametric excitation of oscillators

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Content

Problem statement

Problem:

Numerical study of impedance/frequency small modulation's influence on the oscillator. Analytical solution of ODE set in case of impedance modulation.

For the problem solution it is necessary to analyze ODE set, which follow from the Hamiltonian equation. (2 equation in 1-D case and 4 equation in 2-D). For the numerical solution we used SciLab — a free scientific software package.

Langrangian in general case:

$$L(\mathbf{x}, \dot{\mathbf{x}}, t) = -\frac{1}{2}\mathbf{x}^T \hat{K}\mathbf{x} + \frac{1}{2}\dot{\mathbf{x}}^T \hat{M}\dot{\mathbf{x}} + \mathbf{x}^T \hat{\beta}\dot{\mathbf{x}} \qquad (1)$$

For 1-D case Langrangian and Hamiltonian:

$$L(x, \dot{x}, t) = -\frac{1}{2}x^2 K(t) + \frac{1}{2}m(t)\dot{x}^2 + \beta(t)x\dot{x}$$

$$H(p, x, t) = \frac{1}{2m(t)}(p - x\beta(t))^2 + \frac{1}{2}x^2 K(t)$$

$\hat{K}$ – elastic matrix $\hat{M}$ – mass matrix $\hat{\beta}$ – magnet field matrix

1-D oscillator

With use of reexpression:

$$p' = p - \beta(t)x, \quad K'(t) = K(t) + \frac{\mathrm{d}\beta}{\mathrm{d}t}, \quad x' = x$$

it is possible to change to new canonical variables:

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = p\dfrac{\beta(t)}{m(t)} - \left(K(t) + \dfrac{\beta^2(t)}{m(t)}\right)x \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{p}{m(t)} - \dfrac{\beta(t)}{m(t)}x \end{cases} \Longrightarrow \begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -K(t)x \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{p}{m(t)} \end{cases}$$

Notation

Cyclic frequency (frequency) $\omega(t) \overset{\text{not}}{=} \sqrt{\frac{K(t)}{m(t)}}.$

Impedance $Z(t) \overset{\text{not}}{=} \sqrt{K(t)m(t)}$

With use of new notation, the Hamiltonian equation become:

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\omega(t)Z(t)x(t) \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\omega(t)}{Z(t)}p(t) \end{cases}$$

Case 1

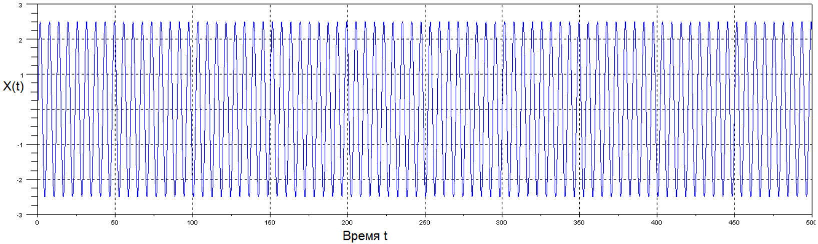
There is no impedance modulation (it is constant), but there is frequency modulation (with $2\omega_0$ frequency).

$$Z(t) = \mathrm{const} = Z_0 \quad \omega(t) = \omega_0 + \omega_1 \cos(2\omega_0 t) \quad \omega_1 << \omega_0$$

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -(\omega_0 + \omega_1 \cos(2\omega_0 t)) Z_0 x(t); \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\omega_0 + \omega_1 \cos(2\omega_0 t)}{Z_0} p(t); \end{cases} \qquad \begin{aligned} p(t_0) &= p_0; \\ x(t_0) &= x_0; \\ t_0 &= 0; \end{aligned}$$

Initial data

$$p_0 = 10; x_0 = 0; \omega_0 = \pi/3; \omega_1 = \omega_0/1000; Z_0 = 4; Z_1 = Z_0/100$$



Conclusion

When there is frequency modulation, but the impedance is constant — there is no parametric resonance.

Case 2

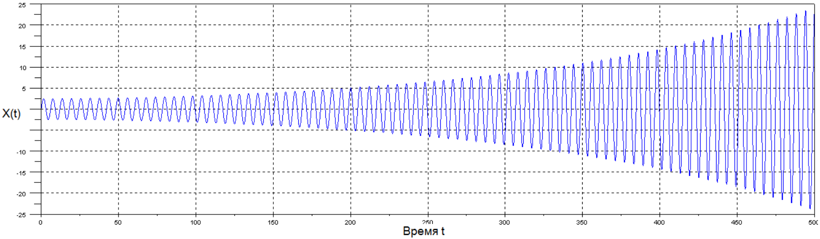
There is no frequency modulation (constant), but there is impedance modulation (with same $2\omega_0$ frequency).

$$\omega(t) = \mathrm{const} = \omega_0 \quad Z(t) = Z_0 + Z_1 \cos(2\omega_0 t) \quad Z_1 << Z_0$$

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}t} = -(Z_0 + Z_1 \cos(2\omega_0 t))\omega_0 x(t), \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\omega_0}{Z_0 + Z_1 \cos(2\omega_0 t)} p(t), \end{cases} \qquad \begin{aligned} p(t_0) &= p_0; \\ x(t_0) &= x_0; \\ t_0 &= 0; \end{aligned}$$

Initial data

$$p_0 = 10; x_0 = 0; \omega_0 = \pi/3; \omega_1 = \omega_0/1000; Z_0 = 4; Z_1 = Z_0/100$$



Conclusion

When there is impedance modulation (even with the constant frequency) — there are parametric resonance.

2-D oscillator

Impedance

In the general case impedance matrix:

$$\hat{Z}\hat{M}^{-1}\hat{Z} = \hat{K}$$

$$\hat{K} = \begin{pmatrix} k_x & 0 \\ 0 & k_y \end{pmatrix} \hat{M} = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \Longrightarrow \hat{Z} = \begin{pmatrix} \sqrt{k_x m} & 0 \\ 0 & \sqrt{k_y m} \end{pmatrix}$$

Hamiltonian:

$$H(\boldsymbol{p}, \boldsymbol{x}, t) = \frac{1}{2m}(p_x^2 + p_y^2) + \frac{1}{2}(Z_x \omega_x x^2 + Z_y \omega_y y^2)$$

Hamiltonian equations in 2-D case:

$$
\begin{cases}
\dfrac{\mathrm{d}p_x}{\mathrm{d}t} = -\omega_x(t) Z_x(t) x(t) \\
\dfrac{\mathrm{d}p_y}{\mathrm{d}t} = -\omega_y(t) Z_y(t) y(t) \\
\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{\omega_x(t)}{Z_x(t)} p_x(t) \\
\dfrac{\mathrm{d}y}{\mathrm{d}t} = \dfrac{\omega_y(t)}{Z_y(t)} p_y(t)
\end{cases}
$$

There are two cases:

- there is frequency modulation, impedance is constant;
- there is impedance modulation, frequency is constant;

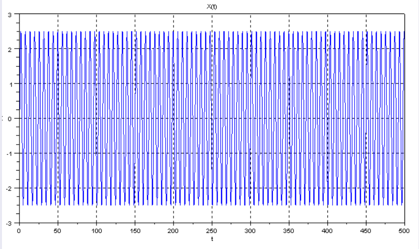
Modulation of $\omega_x(t)$ and $\omega_y(t)$

$$\omega_x(t) = \omega_{0x} + \omega_{1x}\cos(2\omega_{0x}t)$$
$$\omega_y(t) = \omega_{0y} + \omega_{1y}\cos(2\omega_{0y}t)$$
$$Z_x(t) = Z_{0x}$$
$$Z_y(t) = Z_{0y}$$

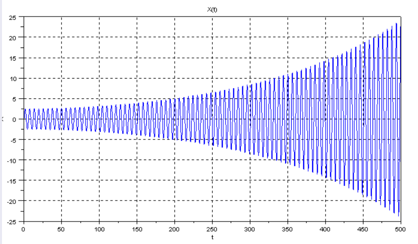
Modulation of $Z_x(t)$ and $Z_y(t)$

$$Z_x(t) = Z_{0x} + Z_{1x}\cos(2\omega_{0x}t)$$
$$Z_y(t) = Z_{0y} + Z_{1y}\cos(2\omega_{0y}t)$$
$$\omega_x(t) = \omega_{0x}$$
$$\omega_y(t) = \omega_{0y}$$

Modulation of $\omega_x(t)$ and $\omega_y(t)$



Modulation of $Z_x(t)$ and $Z_y(t)$



Analytical solution

Eliminating variable p from the Hamiltonian equations (for 1-D oscillator):

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \left(\frac{Z'(t)}{Z(t)} - \frac{\omega'(t)}{\omega(t)}\right)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega^2(t)x(t) = 0. \quad (2)$$

Let $\omega(t) = \omega_0 = \mathrm{const}, \quad Z(t) = Z_0 + Z_1 \cos(2\omega_0 t), Z_1 \ll Z_0$, substitute in (2), we will get:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\omega\frac{Z_1}{Z_0}\sin(2\omega_0 t)\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x(t) = 0.$$

If we use $x(t) = u(t)z(t)$ we will get the equation:

$$\frac{\mathrm{d}^2 z(t)}{\mathrm{d}t^2} + \omega_0^2(1 + h\cos 2\omega_0 t)z(t) = 0, \quad h \overset{\text{not}}{=} 2\frac{Z_1}{Z_0} \ll 1$$

we should try solution of equation:

$$z(t) = a(t)\cos\omega_0 t + b(t)\sin\omega_0 t.$$

substitute in our equation:

$$(a''(t) + 2b'(t)\omega_0 - h\omega_0^2 a(t))\cos\omega_0 t + \\ + (b''(t) - 2\omega_0 a'(t) + h\omega_0^2 b(t))\sin\omega_0 t = 0.$$

Equality to zero is possible only if both multiplier of cos and sin are zero in same time:

$$\begin{cases} a''(t) + 2b'(t)\omega_0 - h\omega_0^2 a(t) = 0, \\ b''(t) - 2\omega_0 a'(t) + h\omega_0^2 b(t) = 0. \end{cases} \tag{3}$$

Try solution:

$$a(t) = k_1 e^{\lambda t}, b(t) = k_2 e^{\lambda t}.$$

We get formula for $a(t)$ and $b(t)$:

$$a(t) = C_1 \operatorname{ch}(\omega_0 A t) + C_2 \operatorname{sh}(\omega_0 A t) + \\ + C_3 \cos(\omega_0 B t) + C_4 \sin(\omega_0 B t),$$

$$b(t) = \left(\frac{2A}{A^2 + h}\right)(C_1 \operatorname{sh} \omega_0 A t + C_2 \operatorname{ch} \omega_0 A t) + \\ + \left(\frac{2B}{B^2 - h}\right)(C_3 \sin \omega_0 B t - C_4 \cos \omega_0 B t).$$

Where

$$A \overset{\text{not}}{=} \sqrt{\sqrt{h^2 + 4} - 2},$$

$$B \overset{\text{not}}{=} \sqrt{\sqrt{h^2 + 4} + 2}.$$

Source

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