On Two-Field Solitons in 2 and 3 Dimensions

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Introduction

- Theoretical description of experimental results obtained at present-day accelerators requires nonperturbative quantum study of the Standard Model (SM)

- SM=QCD+EW[Glashow, Weinberg, Salam]

- Nonperturbative QFTs. Approaches available:

  i) Lattice numerical simulations - from first principles

  ii) Semianalytical techniques, e.g. Dyson-Schwinger (drawback: ambiguities)

  iii) Quantization using localized classical solutions (instantons, dyons, solitons)

$\Rightarrow$ The main problem of solitonic approach - to find stable $3D$ solitons in realistic FTs.
Yet not solved,but not forbidden by theorems, and there is hope...

Gauged NSMs vs EW models

We study 2-field gauge-invariant NSMs (nonlinear sigma models) whose structure resembles that of bosonic sector of EW theory.
In fact, our gauged NSMs include:

i) unit length scalar N-component field, with values on S^{N-1}:

$$s_1{}^2 + ...s_N{}^2 = 1 \qquad (N = 3, 4)$$

interacting with

ii) vector field with $U(1)$ or $SU(2)$ symmetry (Maxwell or Yang-Mills).

Compare with EW bosonic sector -

omitting there $U(1)$ field and freezing radial component of Higgs field one gets $SU(2)$-Higgs frozen model. It containes:

i) unit 4-component scalar field with values on S^3

ii) vector $SU(2)$ Yang-Mills field.

$\Rightarrow$ Clear resemblance !
$\Rightarrow$ Both gauged NSMs and EW theory are invariant with respect to local gauge transformations.

A3M model in 2 dimensions

$$\mathcal{L} = \bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3 - V(s_a) - \frac{1}{4} F_{\mu\nu}^2,$$
$$\bar{\mathcal{D}}_\mu = \partial_\mu + ieA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - ieA_\mu,$$
$$s_+ = s_1 + is_2, \quad s_- = s_1 - is_2,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta(1 - s_3^2),$$

where β, e are coupling constants and $\mu, \nu = 0, 1, 2.$ We look for the topological solitons of the A3M model using the "hedgehog-like" ansatz for the A3-field

$$s_1 = \cos m\chi \sin\theta(R), \quad s_2 = \sin m\chi \sin\theta(R),$$
$$s_3 = \cos\theta(R),$$
$$\sin\chi = \frac{y}{R}, \quad \cos\chi = \frac{x}{R}, \quad R^2 = x^2 + y^2,$$

where m is an integer number, and the standard "vortex" ansatz for the vector field A_μ, describing localized distributions of a stationary magnetic field:

$$A_0 = 0, \quad A_1 = A_x = -ma(R)\frac{y}{R^2},$$
$$A_2 = A_y = ma(R)\frac{x}{R^2}.$$

Topological charge (mapping degree $R^2_{comp} \rightarrow S^2$)

$$Q_t = m$$

Making rescaling

$$a = \alpha e^{-1}, \quad R = re^{-1},$$

A3M (2)

We get for stationary Hamiltonian density $\mathcal{H}(r) = e^{-2}\mathcal{H}(R)$:

$$\mathcal{H}(r) = \left(\frac{d\theta}{dr}\right)^2 + \sin^2\theta \left[p + \frac{m^2(\alpha-1)^2}{r^2}\right] + \frac{m^2}{2}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)^2,$$

$$H = \int_0^\infty \mathcal{H}(r) 2\pi r dr, \quad p = \frac{\beta}{e^2}.$$

Calculating $\delta H/\delta\theta$ and $\delta H/\delta\alpha$, we obtain coupled equations for $\theta(r)$ and $\alpha(r)$:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0,$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} + 2\sin^2\theta(1-\alpha) = 0,$$

to be solved under the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0,$$
$$\alpha(0) = 0, \quad \frac{d\alpha}{dr}(\infty) = 0.$$

A3M (3)

Using series expansion of $\theta(r)$ and $\alpha(r)$ at $r \to 0$, we find from Eqs. (20) and (21) for $m = 1$

$$\theta(r) = \pi - C_1 r + o(r),$$

$$\alpha(r) = r^2(E_1^2 - \frac{1}{4}C_1^2 r^2) + o(r^4).$$

and for $m = 2$

$$\theta(r) = \pi - C_2 r^2 + o(r^2),$$

$$\alpha(r) = r^2(E_2^2 - \frac{1}{12}C_2^2 r^4) + o(r^6).$$

The asymptotic form of the soliton solution for $r \to \infty$ is:

$$\theta(r) \approx \frac{T}{\sqrt{r}} \exp(-\sqrt{p}r), \quad T = const,$$

$$\alpha(r) \approx \alpha_\infty - (1 - \alpha_\infty)\frac{T^2}{2rp} \exp(-2\sqrt{p}r).$$

Solutions exist for $0 < p < p_{cr} \approx 0.41$.

A3M: solutions(4)

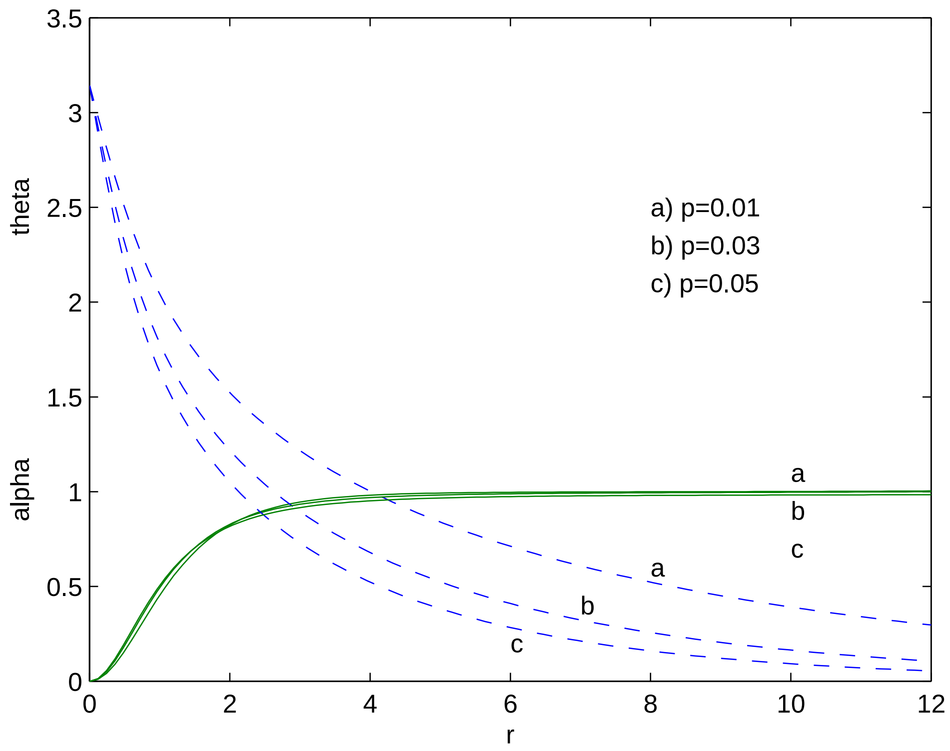

Figure 1: $\theta\ (r)$ and $\alpha\ (r)$ for $p =\ 0.01, 0.03,\ 0.05.$

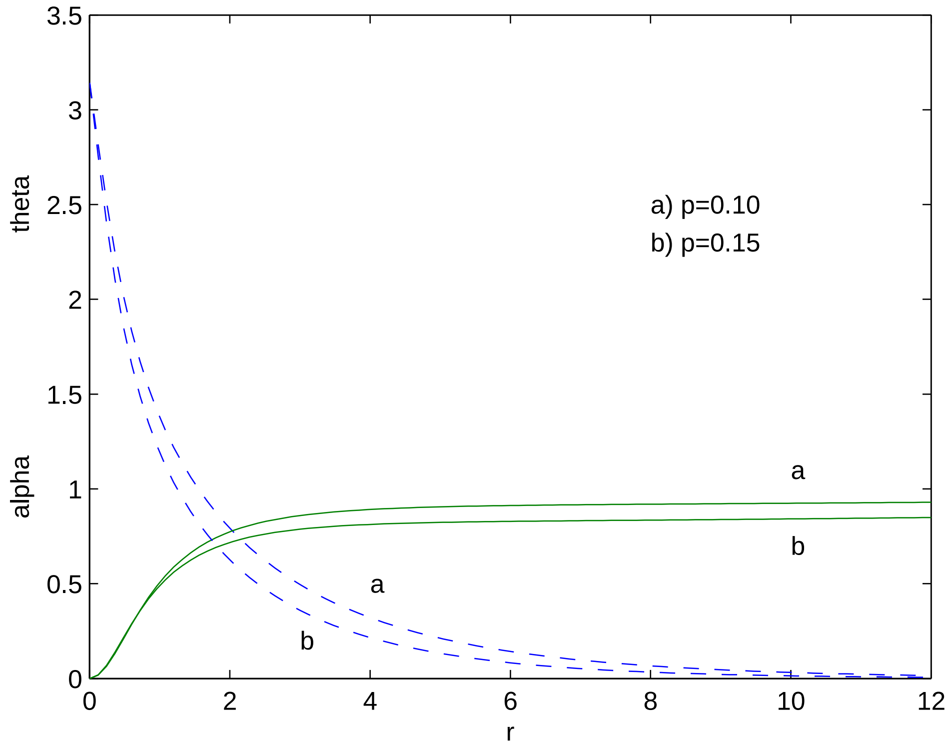

Figure 2: $\theta(r)$ and $\alpha(r)$ for $p = 0.1, 0.15.$

A3M: energy density and $B(r)$

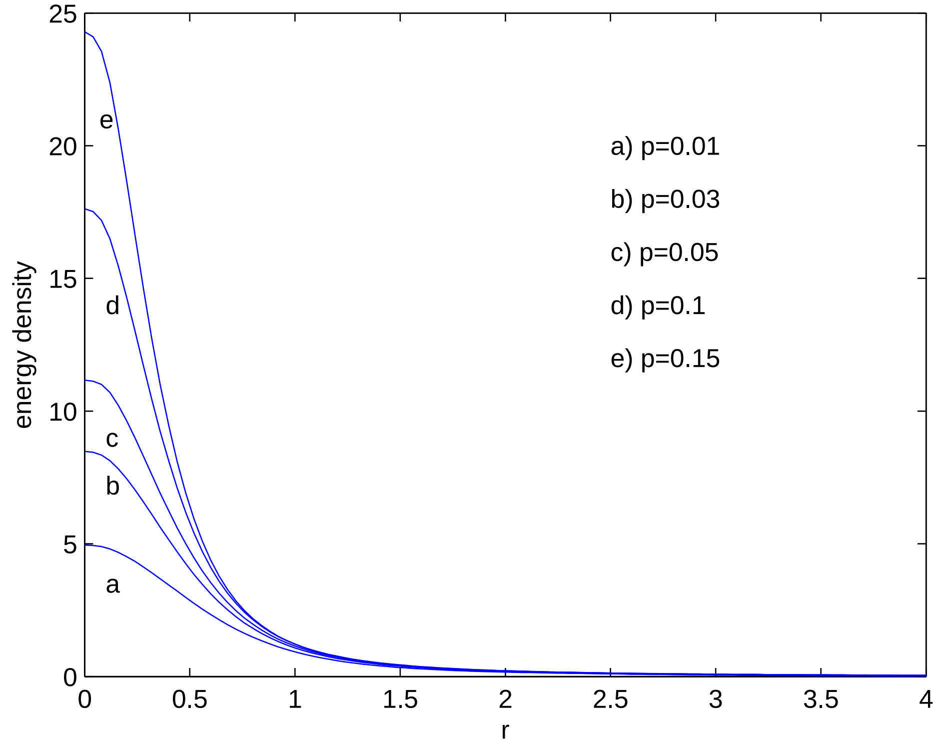

Figure 3: Energy density for various p.

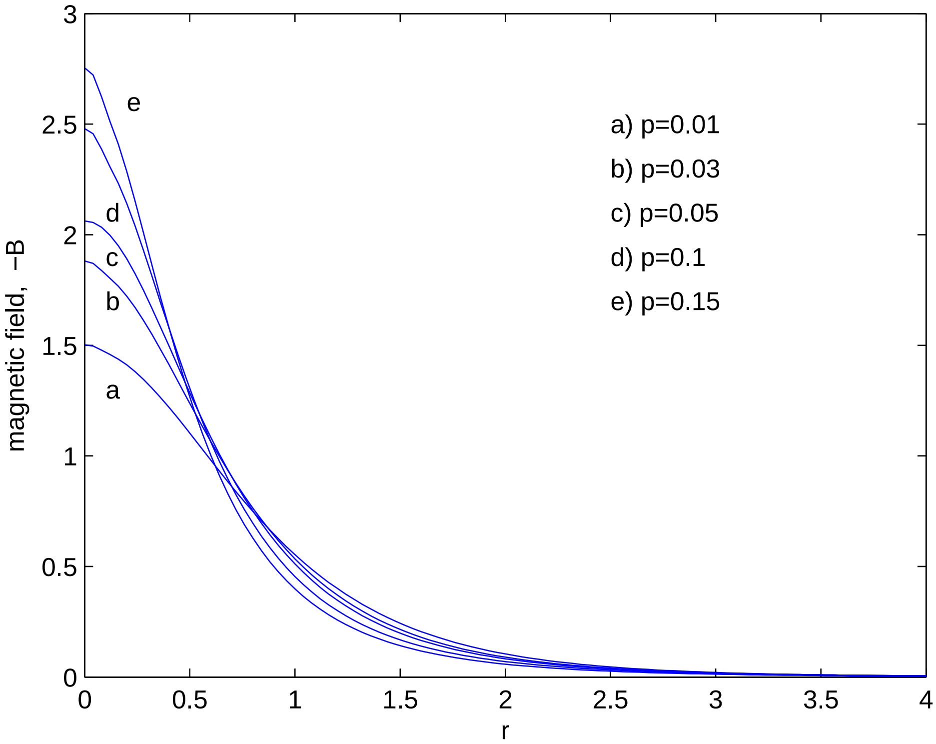

Figure 4: Magnetic field for various p.

A3M: Soliton energy and $\alpha(\infty)$ vs p

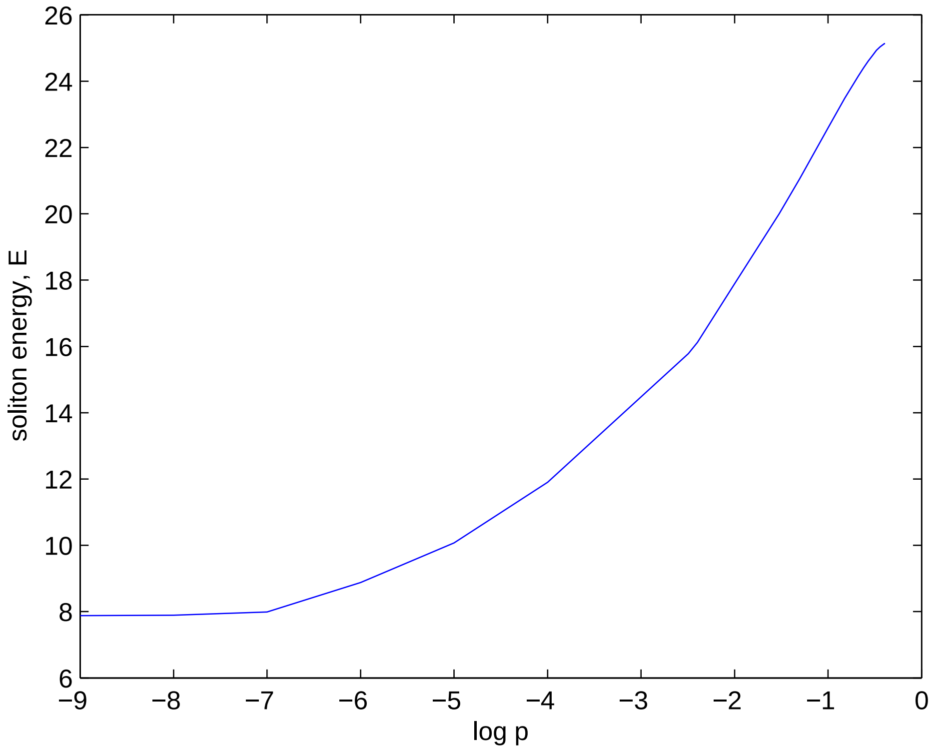

Figure 5: Soliton energy for various p.

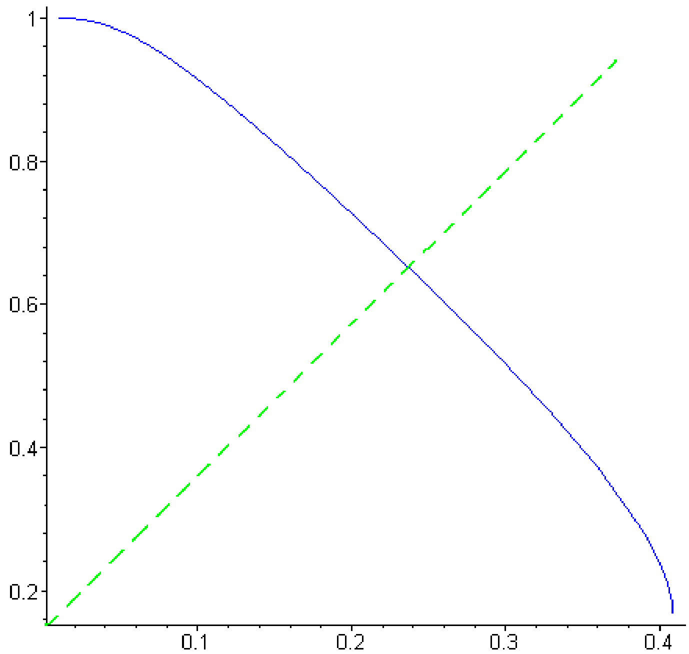

Figure 6: $\alpha(\infty)$ vs p.

$Q_t = 2$ solitons: $m = 2$.
Numerically found, that
$E_{sol}(Q_t = 2, p) < 2 * E_{sol}(Q_t = 1, p)$ for all $0 < p < p_{cr}$,
hence two solitons attract to each other.

CONCLUSIONS

⇒ For $p < p_{cr}$ A3M solitons do exist and prove to be stable

⇒ One can quantize the model around these solitons, quasiclassically (WKB) or in lattice approach.

⇒ One can consider 2D A3M solitons as extended strings embedded into 3D space (e.g., “cosmic strings”).

A4YM model (1)

- Includes unit 4-component scalar field $q^{\alpha}(x^{\mu})$ ("the A4-field") interacting with the vector $SU(2)$ Yang-Mills field $A^{a}_{\mu}(x^{\nu})$

$$\mathcal{L} = \mathcal{D}_{\mu} q^{a} \mathcal{D}^{\mu} q^{a} + \partial_{\mu} q^{0} \partial^{\mu} q^{0} - V(q^{0}) - \frac{1}{4}(F^{a}_{\mu\nu})^{2},$$

$$\mathcal{D}_{\mu} q^{a} = \partial_{\mu} q^{a} + g\varepsilon^{abc} A^{b}_{\mu} q^{c},$$
$$F^{a}_{\mu\nu} = \partial_{\mu} A^{a}_{\nu} - \partial_{\nu} A^{a}_{\mu} + g\varepsilon^{abc} A^{b}_{\mu} A^{c}_{\nu},$$
$$V(q^{0}) = \beta[1 - (q^{0})^{2}],$$

$$\alpha, \mu, \nu = 0, 1, 2, 3, \quad a, b, c = 1, 2, 3, \quad \beta, g-const.$$

- We use ansatzes:

$$q^{0} = \cos\theta(R), \quad q^{a} = \sin\theta(R)\frac{x^{a}}{R}, \quad R^{2} = x^{2}+y^{2}+z^{2},$$

$$A^{a}_{0} = 0, \quad A^{a}_{i} = c(R)\varepsilon^{iak}x^{k}.$$

- Then the Hamiltonian density distributions of localized field bunches are spherically symmetric:

$$\mathcal{H}_{st}(R) = \left(\frac{d\theta}{dR}\right)^{2} + \frac{2\sin^{2}\theta}{R^{2}} + 4gc\sin^{2}\theta + 2g^{2}c^{2}R^{2}\sin^{2}\theta$$

$$+6c^{2} + \left(\frac{dc}{dR}\right)^{2} R^{2} + \frac{1}{2}g^{2}c^{4}R^{4} + 4Rc\frac{dc}{dR} + 2gR^{2}c^{3} + \beta\sin^{2}\theta.$$

A4YM model (2)

Introduce dimensionless variables

$$r = gR, \quad b(r) = g^{-1}cr^2.$$

Calculating $\delta\mathcal{H}/\delta\theta$ and $\delta\mathcal{H}/\delta b$, we get coupled equations ($P = \frac{\beta}{g^2}$)

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{2(b+1)^2}{r^2} + P\right] = 0,$$

$$\frac{d^2b}{dr^2} - \frac{2b}{r^2} - 2\sin^2\theta(1+b) - \frac{b^2}{r^2}(b+3) = 0.$$

We looked for localized solutions, setting the following boundary conditions:

$$\theta(0) = \pi, \quad \theta(\infty) = 0,$$

$$b(0) = 0, \quad b(\infty) = B.$$

Solutions to above problem would define localized distributions $q^\alpha(x^k)$, $\alpha = 0,1,2,3$, $k = 1,2,3$, of the A4-field with unit topological charge, $Q_t = 1$. Here Q_t is the "mapping degree", of continuous maps $R^3_{comp} \to S^3$. However such solutions have not been found. Because of that we look for more general ansatz.

A4YM model (3)

More general ansatz keeps hedgehog form for q^α and generalized ansatz for A_i:

$$A_i^a(x) = \epsilon_{aij} n_j C(r) r + (\delta_{a_i} - n_a n_i)\frac{B(r)}{r} + n_a n_i \frac{G(r)}{r}.$$

However such anzats should respect Coulomb (Lorentz) gauge. Equating $\delta A_i^a/\delta x_i = 0$, we found $C(r) = B(r) + const$. And finally we choose the ansatz

$$A_i^a(x) = \epsilon_{aij} n_j C(r) r + \delta_{a_i}\frac{B(r)}{r} + \frac{G n_a n_i}{r}.$$

$\Rightarrow$ Now calculate hamiltonian density for such ansatz.

Equating variation derivatives $\frac{\delta \mathcal{H}(r)}{\delta C}, \frac{\delta \mathcal{H}(r)}{\delta B}, \frac{\delta \mathcal{H}(r)}{\delta \theta}$ to 0, we obtain coupled equations for radial functions $C(r), B(r), \theta(r)$. Their solution should define localized (solitonic) solutions to A4YM model.

$\Rightarrow$ The study is in progress.

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\begin{aligned}
Hst := {} & \frac{1}{2} R^4 g^2 \mathrm{c}(R)^4 + \left(-2\, g\, \mathrm{c}(R)^3 + 2\, g^2\, \mathrm{c}(R)^2 - 2\, \mathrm{c}(R)^2 g^2 \cos(\theta)^2 + \left(\frac{d}{dR} \mathrm{c}(R) \right)^2 \right) R^2 \\
& + 4\, R \left(\frac{d}{dR} \mathrm{c}(R) \right) \mathrm{c}(R) + 6\, \mathrm{c}(R)^2 + 9\, g^2\, \mathrm{c}(R)^2 G^2 + 18\, g^2\, \mathrm{B}(R)^2 \mathrm{c}(R)^2 + 18\, \mathrm{B}(R)\, G\, g^2\, \mathrm{c}(R)^2 \\
& + \left(\frac{d}{dR} \theta(R) \right)^2 - 4\, g\, \mathrm{c}(R) + 4\, g\, \mathrm{c}(R) \cos(\theta)^2 \\
& - \frac{18\, g \left(\mathrm{B}(R) \left(\frac{d}{dR} \mathrm{c}(R) \right) - \left(\frac{d}{dR} \mathrm{B}(R) \right) \mathrm{c}(R) \right) (\mathrm{B}(R) + G)}{R} + \Bigg(-18\, g^2\, \mathrm{B}(R)^2 \cos(\theta)^2 \\
& + 9 \left(\frac{d}{dR} \mathrm{B}(R) \right)^2 + 2 + 18\, g^2\, \mathrm{B}(R)^2 - 2 \cos(\theta)^2 - 18\, g\, \mathrm{B}(R)^2 c - 18\, g\, \mathrm{c}(R)\, G^2 \\
& - 54\, g\, \mathrm{B}(R)^2 \mathrm{c}(R) - 72\, \mathrm{B}(R)\, G\, g\, \mathrm{c}(R) \Bigg) \Big/ R^2 - \frac{18 \left(\frac{d}{dR} \mathrm{B}(R) \right) (\mathrm{B}(R) + G)}{R^3} \\
& + \frac{162\, G\, \mathrm{B}(R)^3 g^2 + 9\, G^2 + 81\, \mathrm{B}(R)^2 G^2 g^2 + 18\, G\, \mathrm{B}(R) + \frac{243}{2} \mathrm{B}(R)^4 g^2 + 9\, \mathrm{B}(R)^2}{R^4}
\end{aligned}
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